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Infinite Bernoulli convolutions generated by multigeometric series and their properties [1]

Abstract. The paper is devoted to infinite Bernoulli convolutions generated by positive multigeometric series and to probability distributions of random variables whose digits in an even integer base-s expansion with two redundant digits form a sequence of independent and identically distributed random variables.

The main objects of the article are random variables:

$\xi = \sum\limits_{n=1}^{\infty} \frac{\xi_n}{s^n}$, where (ξ_n) is a sequence of independent and identically distributed random variables taking values $0, 1, 2, \ldots, s-1, s, s+1$ with probabilities p_0, p_1, $p_2, \ldots, p_{s-1}, p_s, p_{s+1}$ respectively $(3 < s \in \mathbb{N})$;

$$\eta = \sum_{n=1}^{\infty} \left[\frac{3\eta_{(n-1)(m+1)+1}}{s^n} + \sum_{j=1}^{m} \frac{2\eta_{(n-1)(m+1)+1+j}}{s^n} \right],$$

where (η_n) is a sequence of independent and identically distributed random variables that take values 0 and 1 with probabilities $q_0 > 0$ and $q_1 = 1 - q_0 > 0$. We study conditions under which the above random variables have absolutely continuous or singular distributions as well as topological, metric, and fractal properties of their supports. The main focus is on the case where the spectrum is a Cantorval.

For the case $s = 4$, we establish necessary and sufficient conditions for singularity and absolute continuity of the distributions of ξ and η, in particular when they are supported on the Guthrie–Nymann Cantorval. For an arbitrary even $s > 4$, we determine necessary and sufficient conditions for a random variable ξ to admit a decomposition into the sum of two independent random variables, one of which is uniformly distributed on the unit interval and therefore absolutely continuous. Using the method of characteristic functions, sufficient conditions for singularity and necessary conditions for absolute continuity are obtained. For Cantorvals arising as spectra of the corresponding distributions, the structure and fractal properties of the boundary are investigated.

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Key words: infinite Bernoulli convolutions, singular distributions of random variables, the set of subsums, Cantorval, multigeometric series

Анотація. Стаття присвячена нескінченним згорткам Бернуллі, керованим додатними мультигеометричними рядами, і розподілам випадкових величин, цифри зображення яких у системі з парною натуральною основою s і двома надлишковими цифрами є незалежними та однаково розподіленими випадковими величинами.
Основними об'єктами дослідження є випадкові величини:
$\xi = \sum\limits_{n=1}^{\infty} \frac{\xi_n}{s^n}$, де (ξ_n) – послідовність незалежних випадкових величин, які набувають значень $0, 1, 2, ..., s-1, s, s+1$ з ймовірностями p_0, p_1, $p_2, ..., p_{s-1}, p_s, p_{s+1}$ відповідно $(3 < s \in \mathbb{N})$;

$$\eta = \sum_{n=1}^{\infty} \left[\frac{3\eta_{(n-1)(m+1)+1}}{s^n} + \sum_{j=1}^{m} \frac{2\eta_{(n-1)(m+1)+1+j}}{s^n} \right],$$

де (η_n)– послідовність незалежних однаково розподілених випадкових величин, які набувають значень 0 та 1 з ймовірностями $q_0 > 0$ і $q_1 = 1 - q_0 > 0$ відповідно. Вивчаються умови абсолютної неперервності та сингулярності розподілів цих випадкових величин, топологометричні і фрактальні властивості їх носіїв. Основна увага зосереджена на випадках, коли спектром розподілу є канторвал.
У випадку $s = 4$ для випадкових величин ξ та η знайдено необхідні і достатні умови сингулярності та абсолютної неперервності розподілів, зокрема коли вони зосереджені на канторвалі Гатрі-Німана. Для довільного парного $s > 4$ знайдено необхідні і достатні умови, за яких ξ розкладається в суму двох незалежних випадкових величин, одна з яких має рівномірний розподіл на одиничному відрізку, що рівносильно її абсолютній неперервності. За допомогою методу характеристичних функцій отримано достатні умови сингулярності та необхідні умови абсолютної неперервності. Для канторвалів, що є спектрами розподілів, вивчено структуру і фрактальні властивості межі.
Ключові слова: нескінченні згортки Бернуллі, сингулярні розподіли випадкових величин, множини неповних сум, канторвал, мультигеометричні ряди

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1. Introduction

Singular continuous distributions of random variables are concentrated on sets of Lebesgue measure zero, often with fractal structure. Such distributions naturally arise in the study of random variables with independent digits in various systems of representation (coding) of real numbers [30], [31]. A typical phenomenon is that an infinite Bernoulli convolution generated by a convergent positive series has a singular distribution. We are interested in infinite Bernoulli convolutions generated by positive multigeometric series whose sets of subsums are Cantorvals. Such series and their sets of subsums are currently the subject of active research. The general topological and metric theory of subsums of absolutely convergent series is not yet sufficiently developed and is still at a constructive stage of study [10].

This paper is devoted to probability measures on sets of subsums of multi-geometric series, in particular those that are Cantorvals.

We recall some key definitions and facts that will be used later on, and also formulate the problems that are partially addressed in this paper. Let $A_s = \{0, 1, \dots, s-1\}$ be an alphabet, and let $L_s = A_s \times A_s \times \cdots \times A_s \times \dots$ be the space of sequences with elements from this alphabet for some $2 \le s \in \mathbb{N}$.

A *subsum* of a convergent series $r_0 = v_1 + v_2 + \cdots + v_n + \dots$, determined by a set $M \subset \mathbb{N} = \{1, 2, \dots\}$, is the number

$$x(M) = \sum_{n \in M \subset \mathbb{N}} v_n = \sum_{n=1}^{\infty} \varepsilon_n v_n, \quad \text{where } \varepsilon_n = 1 \text{ if } n \in M, \text{ and } \varepsilon_n = 0 \text{ if } n \notin M.$$

It is clear that each partial sum $S_n = v_1 + v_2 + \cdots + v_n$ and each remainder of the series $r_n = v_{n+1} + v_{n+2} + \dots$ is a subsum of the series, although these do not exhaust all subsums. The *set of subsums* of a given convergent series $\sum v_n$ is the set of all its subsums, i.e.,

$$E(v_n) = \left\{ x : x = \sum_{n=1}^{\infty} \varepsilon_n v_n, \ (\varepsilon_n) \in L_2 \right\}.$$

It is well known that the set of subsums of a convergent positive series is a perfect continuum set that is symmetric with respect to its midpoint $r_0/2$. It may be a finite union of intervals, or it may be a nowhere dense set. Kakeya's result [13] provides necessary and sufficient conditions for the set of subsums to belong to the first type, and sufficient conditions for it to belong to the second type. A complete topological classification of sets of subsums of convergent positive series is presented in [19].

Theorem 1. *The set of subsums $E(v_n)$ of a convergent positive series belongs to one of the following three types:*

1. *an interval or a finite union of intervals;*
2. *a set homeomorphic to the classical Cantor set*

$$C = \left\{ x : x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}, \ \alpha_k \in \{0, 2\} \ \forall k \in \mathbb{N} \right\};$$

3. *an M-Cantorval, that is, a set homeomorphic to*

$$T^* \equiv C \cup \bigcup_{n=1}^{\infty} G_{2n-1} \equiv [0, 1] \setminus \bigcup_{n=1}^{\infty} G_{2n},$$

where

$$G_n = \bigcup_{\alpha_1 \in \{0,1\}} \cdots \bigcup_{\alpha_{n-1} \in \{0,1\}} \left\{ x : \frac{1}{3^n} + \sum_{k=1}^{n-1} \frac{2\alpha_k}{3^k} < x < \frac{2}{3^n} + \sum_{k=1}^{n-1} \frac{2\alpha_k}{3^k} \right\}.$$

At present, necessary and sufficient conditions for the set of subsums to be a Cantorval or a nowhere dense set remain unknown. Criteria for these sets to be of Lebesgue measure zero in the case of nowhere density are not yet fully understood. These problems are being actively investigated for various classes of numerical series with a relatively simple structure [26]. The most significant results in this direction were obtained for multigeometric series

$$k_1 + k_2 + \cdots + k_m + k_1 q + \cdots + k_m q + \cdots + k_1 q^i + \cdots + k_m q^i + \ldots,$$

where $k_1, k_2, \ldots, k_m$ are fixed positive scalars, and $0 < q < 1$. In particular, conditions for the set of subsums of multigeometric series to be a Cantorval were established in [2], [4], [5], [8], [18], with further generalizations provided in [14], [20]. Some conditions for the set of subsums of a non-multigeometric series to be a Cantorval were found in [32].

One of the simplest and most famous Cantorvals was introduced by J. Guthrie and J. Nymann in [11], where the authors considered the set

$$X = \left\{ \sum_{n=1}^{\infty} \frac{\alpha_n}{4^n} : (\alpha_n) \in \{0, 2, 3, 5\}^{\mathbb{N}} \right\},$$

which is the set of subsums of the series

$$\frac{3}{4} + \frac{2}{4} + \frac{3}{4^2} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{2}{4^3} + \cdots + \frac{3}{4^k} + \frac{2}{4^k} + \ldots.$$

Its properties were also studied in [7], [9]. In particular, the latter paper investigates the set U of numbers having a unique representation as a subsum of the Guthrie–Nymann series. It was shown that U is dense in X, but has empty interior. Some modifications of the set X were also studied in [21].

By an *infinite Bernoulli convolution generated by a convergent series* $v_1 + v_2 + \cdots + v_n + \ldots$, we mean the distribution of the random variable $\eta = v_1\eta_1 + v_2\eta_2 + \cdots + v_n\eta_n + \ldots$, where (η_n) is a sequence of independent and identically distributed random variables taking the values 0 and 1 with probabilities $q_0 > 0$ and $q_1 = 1 - q_0 > 0$, respectively.

According to the Jessen–Wintner theorem [12], the distribution of η is either purely absolutely continuous (probability that η takes a value in a zero Lebesgue measure set is 0) or purely singular (there exists a set of Lebesgue measure zero such that η belongs to it with probability 1). In general, necessary and sufficient conditions for an infinite Bernoulli convolution to be singular or absolutely continuous are still unknown. This problem is known as the problem of refining or deepening the Jessen–Wintner theorem. In some special classes of random variables, in particular for infinite Bernoulli convolutions, it can sometimes be solved completely [1], [16], [17], [22], [23], [24], [25], [29].

2. Representations of numbers in a system with a redundant alphabet

Throughout the paper, we use the following notation: m is a fixed positive integer parameter, $s = 2m + 2$, $A = \{0, 1, \ldots, s, s + 1\}$ is the redundant alphabet, $L = A \times A \times \ldots$ is the set of sequences of elements of the alphabet. In a similar manner, we define $A_0 \equiv A \setminus \{1, s\}$ and $L_0 = A_0 \times A_0 \times \cdots$.

If $x = \sum\limits_{n=1}^{\infty} s^{-n}\alpha_n \equiv \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, then the latter symbolic expression will be called the Δ-*representation* of x. Parentheses in the Δ-representation of a number will indicate a period. A *cylinder* of rank m with base $c_1c_2\ldots c_m$ is the set $\Delta_{c_1c_2\ldots c_m}$ of all numbers x having a Δ-representation $x = \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, such that $\alpha_k = c_k$ for $k = \overline{1, m}$. The cylinder $\Delta_{c_1c_2\ldots c_m}$ is the interval $[a_m; b_m]$, where $a_m = \sum\limits_{k=1}^{m} \frac{c_k}{s^k}$, $b_m = a_m + \frac{s+1}{s^m(s-1)}$, and

$$\Delta_{c_1c_2\ldots c_m} = \Delta_{c_1c_2\ldots c_m0} \cup \Delta_{c_1c_2\ldots c_m1} \cup \cdots \cup \Delta_{c_1c_2\ldots c_m[s+1]}.$$

Note that numbers from the interval $\left[0, \frac{s+1}{s-1}\right]$ in the numeration system with base s and two redundant digits s and $s + 1$ may have a unique representation, a finite, countable, or continuum many distinct representations. Moreover, any pair of consecutive digits $(a, s + j)$ in a representation of a number can be replaced, without changing the value of the corresponding series, by the alternative pair $(a + 1, j)$, where $a \in \{0, 1, \ldots, s\}$ and $j \in \{0, 1\}$. Therefore, the number $x = \Delta_{(0s)}$ has a continuum number of distinct representations, since the pairs $(0, s)$ and $(1, 0)$ are interchangeable. The number $x = \Delta_{(1)}$ has countably many representations $x = \Delta_{(1)} = \Delta_{0(s)} = \Delta_{10(s)} = \Delta_{11\ldots10(s)}$.

Assuming that $s \geq 4$, we prove that the number $x_0 = \Delta_{(23)}$ has a unique Δ-representation. Indeed, suppose that x_0 has another representation $\Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ different from the one indicated above. Then there exists a smallest k such that the digit α_k does not coincide with the corresponding digit in the given representation. If $k = 2n - 1$, then by assumption $\alpha_k \neq 2$. However,

$$\Delta_{23\ldots232(2)} = \Delta_{23\ldots231(s+1)} < x_0 < \Delta_{23\ldots233(0)}.$$

Thus, $\alpha_{2n-1} = 2$. If $k = 2n$, then by assumption $\alpha_k \neq 3$. However,

$$\Delta_{23\ldots2323(2)} = \Delta_{23\ldots2322(s+1)} < x_0 < \Delta_{23\ldots2324(0)}.$$

Therefore, $\alpha_{2n} = 3$. Hence, the number x_0 has a unique Δ-representation.

Almost all numbers in the interval $\left[0, \frac{s+1}{s-1}\right]$, in the sense of Lebesgue measure, have continuum many distinct Δ-representations (a complete answer to the question regarding distinct representations can be found in [27]).

Remark 1. Each redundant alphabet, together with the base s, determines its own geometry (topological, metric, and fractal theory). The case of

two redundant digits has its own unique system of cylinder overlaps, which differs from the case of a single redundant digit. Moreover, it includes a classical example – the Guthrie-Nymann series and Cantorval. Knowledge of the "geometry of representation" of numbers significantly contributes to solving problems in probabilistic number theory.

3. A random variable with independent and identically distributed digits

Let us consider a random variable

$$\xi = \sum_{n=1}^{\infty} s^{-n}\xi_n \equiv \Delta_{\xi_1\xi_2\ldots\xi_n\ldots},$$

where (ξ_n) is a sequence of independent random variables taking the values $0, 1, 2, \ldots, s-1, s, s+1$ with probabilities $p_0, p_1, p_2, \ldots, p_{s-1}, p_s, p_{s+1}$, respectively, where $p_i < 1$. It is clear that the set E_ξ of values of the random variable ξ coincides with the interval $\left[0, \frac{s+1}{s-1}\right]$. We are interested in revealing the type of the distribution of ξ and establishing structural, topological, metric, and fractal properties of its spectrum (the set of growth points of the distribution function).

According to the Jessen–Wintner theorem, the random variable ξ has either a purely singular distribution or a purely absolutely continuous distribution. In the general setting, the problem of finding necessary and sufficient conditions under which the distribution is singular is difficult. This paper is devoted to a partial solution of this problem using the method of characteristic functions and the method of distinguishing the absolutely continuous component. This approach was effectively applied in the study of similar objects in [28].

Note that in the case $m = 1$ and $p_1 = 0 = p_4$, the spectrum of the distribution of ξ coincides with the Guthrie–Nymann Cantorval. For arbitrary $m \in \mathbb{N}$, under the condition $p_1 = 0 = p_s$, the spectrum of the distribution of ξ coincides with the set of subsums of the series

$$\frac{3}{s} + \underbrace{\frac{2}{s} + \cdots + \frac{2}{s}}_{m} + \frac{3}{s^2} + \underbrace{\frac{2}{s^2} + \cdots + \frac{2}{s^2}}_{m} + \cdots + \frac{3}{s^n} + \underbrace{\frac{2}{s^n} + \cdots + \frac{2}{s^n}}_{m} + \ldots,$$

which is also a Cantorval and was partially studied in [3]. The infinite Bernoulli convolution generated by this series is closely related to the distribution of the random variable ξ. We describe this relationship in detail below.

4. Absolutely continuous distributions on the Guthrie–Nymann Cantorval

Theorem 2. *Let $m = 1$, that is, $s = 4$. If $p_2 = p_3 = p_0 + p_4 = p_1 + p_5 = \frac{1}{4}$ and $p_1 \geq p_0$, then ξ has an absolutely continuous distribution.*

Proof. We prove that under the assumptions of the theorem, ξ can be represented as the sum of two independent random variables τ and η, one of which has an absolutely continuous (in fact, uniform) distribution on the interval $[0,1]$. Then, by a well-known fact [15], the sum $\tau+\eta$ also has an absolutely continuous distribution.

Consider the random variable $\tau = \sum\limits_{n=1}^{\infty} 4^{-n}\tau_n$, where (τ_n) is a sequence of independent random variables taking the values $0,1,2,3$ with probability $\frac{1}{4}$ each (as is well known, τ has the uniform distribution on $[0,1]$), and the random variable $\eta = \sum\limits_{n=1}^{\infty} 4^{-n}\eta_n$, where (η_n) is a sequence of independent random variables taking the values $0,1,2$ with probabilities u, v, and $1-u-v$, respectively (it has a singular distribution on $[0,1]$). Then their sum $\tau+\eta=\theta=\sum\limits_{n=1}^{\infty} 4^{-n}\theta_n$ is a random variable whose digits θ_n in the Δ-representation take the values $0,1,2,3,4,5$ with probabilities $p_0=\frac{1}{4}u, p_1=\frac{1}{4}u+\frac{1}{4}v, p_2=\frac{1}{4}=p_3, p_4=\frac{1}{4}-p_0, p_5=\frac{1}{4}-p_1$. Since under the assumptions of the theorem this system is consistent (it has a nonnegative solution with respect to u and v), we obtain that $\xi=\tau+\eta$ has an absolutely continuous distribution.

Corollary 1. *If one of the following conditions holds: $p_0=0=p_1, p_0=0=p_5, p_4=0=p_5$, then ξ has an absolutely continuous distribution.*

The above results can easily be extended to the case of an arbitrary integer $s>3$.

Theorem 3. *The random variable ξ can be represented as the sum of two independent random variables, one of which is uniformly distributed on the interval $[0,1]$, if and only if*

$$p_1 \ge p_0 \quad \text{and} \quad p_2=p_3=\cdots=p_{s-2}=p_{s-1}=p_0+p_s=p_1+p_{s+1}=\frac{1}{s}. \tag{4.1}$$

Proof. Let us consider a random variable $\tau = \sum\limits_{n=1}^{\infty} s^{-n}\tau_n$, where (τ_n) is a sequence of independent random variables taking the values $0,1,\ldots,s-1$ with probability $\frac{1}{s}$ each (as is well known, τ has the uniform distribution on $[0,1]$), and the random variable $\eta = \sum\limits_{n=1}^{\infty} s^{-n}\eta_n$, where (η_n) is a sequence of independent random variables taking the values $0,1,2$ with probabilities u, v, and $1-u-v$, respectively (it has a singular Cantor-type distribution on $[0,1]$). Then their sum $\tau+\eta=\theta=\sum\limits_{n=1}^{\infty} s^{-n}\theta_n$ is a random variable whose digits θ_n in the Δ-representation take the values $0,1,\ldots,s,s+1$ with probabilities $p_0=\frac{1}{s}u, p_1=\frac{1}{s}u+\frac{1}{s}v, p_2=\frac{1}{s}=p_3=\cdots=p_{s-1}, p_s=\frac{1}{s}-p_0, p_{s+1}=\frac{1}{s}-p_1$. Since under the assumptions of the theorem this system is consistent (it has a solution with respect to u and v), we obtain that $\xi=\tau+\eta$ has an absolutely continuous distribution.

5. Conditions for singularity of the distribution: the method of characteristic functions

To obtain sufficient conditions for the singularity of the distribution of ξ, we use the method of characteristic functions. Recall that the *characteristic function* of the distribution of a random variable ζ with distribution function $F_\zeta(x)$ is the complex-valued function $f_\zeta(t)$ defined as the expectation of the random variable $e^{it\zeta}$, i.e.,

$$f_\zeta(t) = Me^{it\zeta} = \int\limits_{-\infty}^{+\infty} e^{itx}\, dF_\zeta(x).$$

It is known [15] that for an absolutely continuous distribution of a random variable ζ with characteristic function $f_\zeta(t)$, the number $L_\zeta \equiv \lim\limits_{|t|\to\infty} \sup |f_\zeta(t)|$ is equal to 0; for a discrete distribution, $L_\zeta = 1$; and for a singular distribution $L_\zeta \in [0, 1]$. For a singular distribution, the value of L_ζ characterizes its closeness to either an absolutely continuous or a discrete distribution. For a continuous random variable ζ with a pure Lebesgue type distribution, the condition $L_\zeta \neq 0$ implies singularity. This is the essence of the method of characteristic functions for proving singularity of a distribution.

Since (ξ_n) is a sequence of independent random variables and $\xi = \frac{\xi_1}{s} + \frac{\hat{\xi}}{s}$, where $\hat{\xi} = \frac{\xi_2}{s} + \frac{\xi_3}{s^2} + \ldots$, we conclude that ξ_1 and $\hat{\xi}$ are independent, and $\hat{\xi}$ has the same distribution as ξ. Thus, for the characteristic functions of ξ and $\hat{\xi}$ we have $f_\xi(t) = f_{\hat{\xi}}(t)$. By definition, we get

$$f_\xi(t) = Me^{it\xi} = Me^{it\sum\limits_{k=1}^{\infty} s^{-k}\xi_k} = Me^{it\frac{\xi_1}{s}} \cdot Me^{i\frac{t}{s}\hat{\xi}} = \varphi_1(t) f_\xi\left(\frac{t}{s}\right),$$

where

$$\varphi_1(t) = Me^{it\frac{\xi_1}{s}} = p_0 + p_1 e^{i\frac{t}{s}} + p_2 e^{i\frac{t}{s}\cdot 2} + \cdots + p_{s+1} e^{i\frac{t}{s}(s+1)}.$$

Therefore, if $\varphi_1(t_0) = 1$, then $f_\xi(t_0) = f_\xi\left(\frac{t_0}{s}\right)$.

Since $\varphi_1(2\pi s^n) = 1$, it follows that

$$f_\xi(2\pi) = f_\xi(2\pi s) = \cdots = f_\xi(2\pi s^{n-1}) = f_\xi(2\pi s^n), \qquad n \in \mathbb{N}.$$

Now we have proved the following statement.

Lemma 1. *The characteristic function $f_\xi(t)$ of the random variable $\xi = \Delta_{\xi_1\xi_2\ldots\xi_n\ldots}$ admits the representation*

$$f_\xi(t) = \lim_{m\to\infty} \prod_{k=1}^{m} \varphi_k(t) = \prod_{k=1}^{\infty} \varphi_k(t), \qquad \text{where } \varphi_k(t) = \sum_{m=0}^{s+1} p_m e^{i\frac{t}{s^k}m},$$

and for every $n \in \mathbb{N}$ satisfies the relation $f_\xi(2\pi) = f_\xi(2\pi s^n)$.

Corollary 2. *The following inequality holds*

$$L_\xi \equiv \lim_{|t|\to\infty} \sup |f_\xi(t)| \geq |f_\xi(2\pi)|.$$

Theorem 4. *If $m = 1$, that is, $s = 4$, and at least one of the conditions $p_2 \neq p_0 + p_4$ or $p_3 \neq p_1 + p_5$ holds, then ξ has a singular distribution on $\left[0, \frac{5}{3}\right]$.*

Proof. We apply the method of characteristic functions. For $s = 4$ we have

$$\varphi_k(t) = \left(p_0 + \sum_{m=1}^{5} p_m \cos \frac{t}{4^k} m\right) + i \sum_{m=1}^{5} p_m \sin \frac{t}{4^k} m,$$

and $\varphi_k(2\pi) \neq 0$ for $k > 1$, while $\varphi_1(2\pi) = (p_0 - p_2 + p_4) + i(p_1 - p_3 + p_5)$. Thus, $\varphi_1(2\pi) = 0$ is equivalent to $p_2 = p_0 + p_4, p_3 = p_1 + p_5$. Next we will show that if these conditions fail, then $|f_\xi(2\pi)| \neq 0$, and hence $L_\xi > 0$.

Consider the product

$$\prod_{k=2}^{\infty} \varphi_k(2\pi). \tag{5.1}$$

Its convergence is equivalent to the convergence of the series

$$\sum_{k=2}^{\infty} |\varphi_k(2\pi) - 1|. \tag{5.2}$$

Let us estimate a general term of this series

$$\begin{aligned}
|\varphi_k(2\pi) - 1| &= \left[\left(p_0 - 1 + \sum_{j=1}^{5} p_j \cos \frac{2\pi}{4^k} j\right)^2 + \left(\sum_{j=1}^{5} p_j \sin \frac{2\pi}{4^k} j\right)^2\right]^{\frac{1}{2}} \leq \\
&\leq 2 \left(\sum_{j=0}^{5} p_j \sin^2 \frac{\pi j}{4^k} - \sum_{r\neq j}^{5} p_r p_j \sin^2 \frac{\pi(r-j)}{4^k}\right)^{\frac{1}{2}} \leq \\
&\leq 2 \left(\sum_{j=0}^{5} p_j \sin^2 \frac{\pi j}{4^k}\right)^{\frac{1}{2}} \leq 2 \left(\sum_{j=0}^{5} p_j \sin^2 \frac{5\pi}{4^k}\right)^{\frac{1}{2}} \leq \\
&\leq 2 \sin \frac{5\pi}{4^k} \leq 2\pi \frac{5}{4^k}.
\end{aligned}$$

Note that the last two inequalities hold for all $k > 1$. Since the geometric series with general term $2\pi \frac{5}{4^k}$ converges, it follows from the comparison test that the series (5.2) also converges, and hence the product (5.1) converges as well. Thus, under the assumptions of the theorem we have $|f_\xi(2\pi)| > 0$ and $L_\xi > 0$. Therefore, ξ has a singular distribution.

Corollary 3. *If $p_1 = p_4 = 0$ and the assumption of the theorem is fulfilled, then ξ has a singular distribution on the Guthrie–Nymann Cantorval.*

Indeed, the theorem implies the singularity of the distribution of ξ, while its spectrum is the Guthrie–Nymann Cantorval.

Theorem 5. *If $q_0 \neq \frac{1}{2}$, then the infinite Bernoulli convolution generated by the Guthrie–Nymann series has a purely singular distribution.*

Proof. Consider the distribution of the random variable

$$\zeta = \frac{3\zeta_1}{4} + \frac{2\zeta_2}{4} + \frac{3\zeta_3}{4^2} + \frac{2\zeta_4}{4^2} + \cdots + \frac{3\zeta_{2k-1}}{4^k} + \frac{2\zeta_{2k}}{4^k} + \ldots,$$

where (ζ_n) is a sequence of independent random variables taking the values 0 and 1 with probabilities q_0 and q_1, respectively. Then the random variable $\eta_k = 3\zeta_{2k-1} + 2\zeta_{2k}$ takes the values $0, 2, 3, 5$ with probabilities q_0^2, q_0q_1, q_0q_1, and q_1^2, respectively. The independence of (ζ_n) implies the independence of the sequence (η_k). Therefore, the distribution of ζ is in fact equivalent to the distribution of ξ under the conditions $p_0 = q_0^2, p_2 = q_0q_1, p_3 = q_0q_1, p_5 = q_1^2, p_1 = 0 = p_4$.

Since the assumption of Theorem 4 is fulfilled, i.e., $q_0q_1 = p_2 \neq p_0 = q_0^2$ or $q_0q_1 = p_3 \neq p_5 = q_1^2$, we conclude that the distribution of ζ is singular whenever $q_0 \neq \frac{1}{2}$ ($q_1 = 1 - q_0 \neq \frac{1}{2}$).

6. The general case

Theorem 6. *Let $m > 1$ and $s = 2m + 2$. If at least one of the following conditions*

$$u \equiv (p_0 + p_s - p_{\frac{s}{2}}) + (p_1 + p_{s+1} + p_{s-1}) \cos\frac{2\pi}{s} + \sum_{j=2}^{m} (p_j + p_{s-j}) \cos\frac{\pi j}{m+1} \neq 0,$$

$$v \equiv (p_1 + p_{s+1} - p_{s-1}) \sin\frac{\pi}{m+1} + \sum_{j=2}^{m} (p_j - p_{s-j}) \sin\frac{\pi j}{m+1} \neq 0,$$

is fulfilled, then $L_\xi \geq |f_\xi(2\pi)| > 0$, and hence the distribution of ξ is singular. On the other hand, if $u = 0 = v$, then $|f_\xi(2\pi)| = 0$, and the question about the type of the distribution of ξ remains open.

Proof. Since the infinite product $\prod\limits_{k=2}^{\infty} \varphi_k(2\pi)$ converges, we have $f_\xi(2\pi) = 0$ if and only if $\varphi_1(2\pi) = 0$. However,

$$\varphi_1(2\pi) = p_0 + \sum_{j=1}^{s+1} p_j \cos\frac{2\pi j}{s} + i \sum_{j=1}^{s+1} p_j \sin\frac{2\pi j}{s} = u + iv,$$

where

$$u = p_0 + \sum_{j=1}^{s+1} p_j \cos\frac{\pi j}{m+1} =$$

$$= (p_0 + p_s - p_{\frac{s}{2}}) + (p_1 + p_{s+1} + p_{s-1}) \cos \frac{\pi}{m+1} + \sum_{j=2}^{m} (p_j + p_{s-j}) \cos \frac{\pi j}{m+1},$$

$$v = \sum_{j=1}^{s+1} p_j \sin \frac{2\pi j}{s} = (p_1 + p_{s+1} - p_{s-1}) \sin \frac{\pi}{m+1} + \sum_{j=2}^{m} (p_j - p_{s-j}) \sin \frac{\pi j}{m+1},$$

because

$$\cos \frac{\pi(s-j)}{m+1} = \cos \left(2\pi - \frac{j\pi}{m+1} \right) = \cos \frac{j\pi}{m+1},$$
$$\sin \frac{\pi(s-j)}{m+1} = \sin \left(2\pi - \frac{j\pi}{m+1} \right) = -\sin \frac{j\pi}{m+1}.$$

Thus, if $u \neq 0$ or $v \neq 0$, we have $\varphi_1(2\pi) \neq 0$ and $L_\xi \geq |f_\xi(2\pi)| > 0$.

If, however, $u = 0 = v$, then $\varphi_1(2\pi) = 0$, and no conclusion about L_ξ can be drawn.

Corollary 4. *A necessary condition for the absolute continuity of the distribution of ξ is $u = 0 = v$.*

Corollary 5. *If $p_1 = 0 = p_s$, $p_0 p_2 p_3 \dots p_{s-1} p_{s+1} \neq 0$, and the assumptions of the theorem are satisfied, then the random variable ξ has a singular distribution whose spectrum is a Cantorval.*

7. Fractal properties of the boundary of the spectrum

Let us consider the set $E_s = \{x : x = \sum_{n=1}^{\infty} \varepsilon_n s^{-n},\ (\varepsilon_n) \in L_0\}$, which is the spectrum of the random variable ξ under the conditions $p_1 = 0 = p_s$ and $p_0 p_2 p_3 \cdots p_{s-1} p_{s+1} \neq 0$. The topological and metric properties of the set E_s, for $s = 2m + 2$, $m \in \mathbb{N}$, were studied in [3]. It was proved there that E_s is a Cantorval and its Lebesgue measure is equal to 1. This measure coincides with the total length of the intervals whose union is the interior of E_s.

Below we extend these results to the case of arbitrary $s \geq 4$, $s \in \mathbb{N}$, and determine the fractal dimension of the boundary of the set E_s.

It is easy to see that E_s is the attractor of an iterated function system (IFS) consisting of s similarities $w_i(x) = \dfrac{i}{s} + \dfrac{x}{s}$ for $i \in A_0$ (see [6]), and hence

$$E_s = \bigcup_{i \in A_0} w_i(E_s) = W(E_s).$$

Lemma 2. *The following properties of $W = \{w_0, w_2, w_3, \dots, w_{s-1}, w_{s+1}\}$ are easy to verify:*

- *The interval $H \equiv \left[0, \frac{s+1}{s-1}\right]$ is the minimal interval containing E_s;*
- $\min w_0(E_s) = 0,\ \max w_0(E_s) = \frac{1}{s} \cdot \frac{s+1}{s-1}$;

- $\min w_i(E_s) = \frac{i}{s}$, $\max w_i(E_s) = \frac{i}{s} + \frac{1}{s} \cdot \frac{s+1}{s-1}$ *for all* $2 \le i \le s-1$;
- $\min w_{s+1}(E_s) = \frac{s+1}{s}$, $\max w_{s+1}(E_s) = \frac{s+1}{s} + \frac{1}{s} \cdot \frac{s+1}{s-1}$.

Note that the distances between cylinders satisfy

$$\rho(\Delta_0, \Delta_2) = \rho(\Delta_{[s-1]}, \Delta_{[s+1]}) = \frac{s-3}{s(s-1)} > 0.$$

Theorem 7. *The interval* $I = \left[\frac{2}{s-1}, 1\right]$ *is contained in the set* E_s *and is the longest interval entirely contained in* E_s. *Moreover,* $\frac{2}{s-1} = \Delta_{(2)} = \Delta^s_{(2)}, 1 = \Delta_{(s-1)} = \Delta^s_{(s-1)}$.

Proof. Recall that every number $x \in [0,1]$ has a classical base-s expansion

$$x = \Delta^s_{a_1 a_2 \ldots a_n \ldots} = \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_n}{s^n} + \ldots, \qquad a_n \in \{0, 1, 2, \ldots, s-1\}. \quad (7.1)$$

We first show that the interval $\left[\frac{3}{s}, 1\right]$ is entirely contained in E_s. Let we have an arbitrary $x \in \left[\frac{3}{s}, 1\right]$. Then it has a base-$s$ expansion $x = \Delta^s_{3a_2 \ldots a_n \ldots}$. Next we show that there exists a sequence $(b_n) \in L_0$ such that

$$x = \Delta^s_{3a_2 \ldots a_n \ldots} = \frac{b_1}{s} + \frac{b_2}{s^2} + \cdots + \frac{b_n}{s^n} + \cdots = \Delta_{b_1 b_2 \ldots b_n \ldots}.$$

If the sequence (a_n) contains no digit 1, then we simply set $b_1 = 3$ and $b_n = a_n$ for $n > 1$. In the case where the sequence (a_n) contains some digits equal to 1, we describe alternative replacements of digit blocks that produce another representation of the same number using only digits from the alphabet A_0. We begin with the case of replacing the period (1) in the representation of a number. It is clear that $x = \Delta^s_{3a_2 \ldots a_m (1)} = \Delta_{3a_2 \ldots a_m (0[s+1])}$.

From now on, we assume that the representation of x does not end with the period (1).

1. If $a_2 = 1$, then the following substitutions eliminate digit 1 in the representation:

$$\Delta^s_{311 \ldots 1 a_{2k+2} a_{2k+3} \ldots} = \Delta_{30[s+1]0[s+1] \ldots 0[s+1] a_{2k+2} a_{2k+3} \ldots},$$

$$\Delta^s_{311 \ldots 1 a_{2k+1} a_{2k+2} \ldots} = \Delta_{2[s+1]0[s+1]0[s+1] \ldots 0[s+1] a_{2k+1} a_{2k+2} \ldots}.$$

2. If $a_2 \ne 1$, then we have

$$\Delta^s_{301 a_4 a_5 \ldots} = \Delta_{2[s-1][s+1] a_4 a_5 \ldots}, \qquad \Delta^s_{3 a_2 1 a_4 a_5 \ldots} = \Delta_{3[a_2 - 1][s+1] a_4 a_5 \ldots}, \quad \text{if } a_2 \ne 0.$$

The following substitutions eliminate the first occurrence of the digit 1 in the representation:

$\Delta^s_{3a_2 \ldots a_m 1 a_{m+2} a_{m+3} \ldots} = \Delta_{3a_2 \ldots a_{m-1}[a_m - 1][s+1] a_{m+2} a_{m+3} \ldots}$;
$\Delta^s_{3a_2 \ldots a_m 01 a_{m+3} a_{m+4} \ldots} = \Delta_{3a_2 \ldots a_{m-1}[a_m - 1][s-1][s+1] a_{m+3} a_{m+4} \ldots}$, $a_m \ne 0$;
$\Delta^s_{3a_2 \ldots a_k 0 \ldots 01 a_{k+1+p} a_{k+2+p} \ldots} = \Delta_{3a_2 \ldots a_{k-1}[a_k - 1][s-1] \ldots [s-1][s+1] a_{k+1+p} \ldots}$, $a_k \ne 0$;

$\Delta^s_{30\ldots01a_{k+3}a_{k+4}\ldots} = \Delta_{2[s-1][s-1]\ldots[s-1][s+1]a_{k+3}a_{k+4}\ldots}$.

After eliminating the first block with 1, the second block becomes the first one, and we proceed by the same algorithm. Continuing inductively, we conclude that for every $x \in \left[\frac{3}{s}, 1\right]$ there exists a representation that does not use the digits 1 and s. Therefore, $\left[\frac{3}{s}, 1\right] \subset E_s$.

For $s \geq 4$ the inequality $\frac{3}{s} < \frac{s+1}{2(s-1)} < 1$ holds, and hence $\left[\frac{s+1}{2(s-1)}, 1\right] \subset \left[\frac{3}{s}, 1\right] \subset E_s$. Since E_s is symmetric with respect to its midpoint $\frac{s+1}{2(s-1)}$, it follows that

$$\left[\frac{s+1}{2(s-1)} - \left(1 - \frac{s+1}{2(s-1)}\right), 1\right] = \left[\frac{2}{s-1}, 1\right] \subset E_s.$$

We now prove that this interval is maximal among all intervals contained in E_s. To do so, we show that every left neighborhood $(\Delta^s_{(2)} - \varepsilon; \Delta^s_{(2)})$ of the point $\Delta^s_{(2)}$ contains an interval disjoint from E_s. Since $\Delta_0 = \left[0, \frac{1}{s(s-1)}\right], \Delta_2 = \left[\frac{2}{s}, \frac{2}{s} + \frac{1}{s(s-1)}\right]$, and $\Delta_0 \cap \Delta_2 = \varnothing$, we obtain $\left(\frac{1}{s(s-1)}, \frac{1}{s}\right) \cap E_s = \varnothing$.

Since $\Delta_{\underbrace{2\ldots20}_{k}} = \left[a_k, a_k + \frac{1}{s^k(s-1)}\right]$, where $a_k = \sum\limits_{i=1}^{k-1} \frac{2}{s^i}$, and $\Delta_{\underbrace{2\ldots20}_{k}} \cap \Delta_{\underbrace{2\ldots20}_{k+1}} = \varnothing$, we get $\left(a_k + \frac{1}{s^k(s-1)}, a_k + \frac{1}{s^k}\right) \cap E_s = \varnothing$. Since $a_k \to \frac{2}{s-1}$ as $k \to \infty$, it is easy to choose $k(\varepsilon)$ such that the gap $\left(a_k + \frac{1}{s^k(s-1)}, a_k + \frac{1}{s^k}\right)$ of E_s lies entirely inside the prescribed left neighborhood of the point $\frac{2}{s-1}$. By symmetry, an analogous situation occurs to the right of 1, namely there exist gaps symmetric to the above ones with respect to the point $\frac{s+1}{2(s-1)}$.

Corollary 6. *The following equality holds:*

$$I = \Delta_3 \cup \Delta_4 \cup \cdots \cup \Delta_{[s-2]} \cup \left[\bigcup_{k=1}^{\infty} \Delta_{\underbrace{2\ldots2}_{k}3}\right] \cup \left[\bigcup_{k=2}^{\infty} \Delta_{\underbrace{[s-1]\ldots[s-1]}_{k}[s-2]}\right].$$

Theorem 8. *The set $E_s \setminus intI$ is a countable union of pairwise disjoint affine copies of E_s, namely,*

$$E_s = \left(\frac{2}{s-1}, 1\right) \cup \bigcup_{i=0}^{\infty} [\varphi_i(E_s) \cup h(\varphi_i(E_s))], \tag{7.2}$$

where $\varphi_i(x = \Delta_{\alpha_1\alpha_2\cdots\alpha_n\cdots}) = \Delta_{\underbrace{2\ldots2}_{i}0\alpha_1\alpha_2\cdots\alpha_n\cdots}$ is the right-shift operator on digits, and $h(x = \Delta_{\alpha_1\alpha_2\cdots\alpha_n\cdots}) = \Delta_{[s+1-\alpha_1]\cdots[s+1-\alpha_n]\cdots} = \frac{s+1}{s-1} - x$ is the digit inversion of the Δ-representation.

Proof. Note that $\varphi_i(x)$ is a similarity transformation with ratio $k_i = \frac{1}{s^{i+1}}$, and hence E_s and $\varphi_i(E_s)$ are similar. The mapping $h(x)$ is the central symmetry with respect to the point $\frac{s+1}{2(s-1)}$ of the set E_s. Therefore, $\varphi_i(E_s)$ and $h(\varphi_i(E_s))$ are isometric figures.

Taking into account the consequences of the previous lemma and theorem, we have

$$\varphi_i(E_s) = \Delta_{\underbrace{2\ldots20}_{i}} \cap E_s, \qquad h(\varphi_i(E_s)) = \Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i}} \cap E_s.$$

The set $E_s \subset H$ is the attractor of the IFS W, and by Lemma 2,

$$w_0(H) \cap w_i(H) = \varnothing, \qquad w_i(H) \cap w_{s+1}(H) = \varnothing \quad \text{for } 2 \le i \le s;$$

$$w_0(H) \cap I = \varnothing, \qquad w_s(H) \cap I = \varnothing.$$

Using again the consequences of the previous lemma and theorem, together with the identities

$$\varphi_i(E_s) = \Delta_{\underbrace{2\ldots20}_{i}} \cap E_s, \qquad h(\varphi_i(E_s)) = \Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i}} \cap E_s,$$

and

$$\rho(\Delta_{\underbrace{2\ldots20}_{i}}, \Delta_{\underbrace{2\ldots20}_{i+1}}) = \frac{s-2}{s^i(s-1)} > 0,$$

we obtain (7.2).

Theorem 9. *The interior $intE_s$ of E_s has Lebesgue measure equal to* 1.

Proof. It is well known that if the sets D_1 and D_2 are similar with similarity ratio k, then their one-dimensional Lebesgue measures satisfy $\lambda(D_2) = k\lambda(D_1)$. Taking into account Theorem 8, and equality (7.2) in particular, we obtain

$$\lambda(\operatorname{int} E_s) = \frac{s-3}{s-1} + \frac{2}{s}\lambda(intE_s) + \frac{2}{s^2}\lambda(intE_s) + \frac{2}{s^3}\lambda(intE_s) + \cdots =$$

$$= \frac{s-3}{s-1} + \frac{2}{s-1}\lambda(intE_s).$$

Hence $\left(1 - \frac{2}{s-1}\right)\lambda(\operatorname{int} E_s) = \frac{s-3}{s-1}$, that implies $\lambda(\operatorname{int} E_s) = 1$.

Theorem 10. *The boundary frE_s of the Cantorval E_s is an N-self-similar set with Hausdorff–Besicovitch dimension* $\log_s 3$.

Proof. The boundary $frE_s = E_s \setminus intE_s$ of the set E_s is an N-self-similar set (it is a union of pairwise disjoint affine copies of itself) since

$$frE_s = \bigcup_{i=1}^{\infty} [\Delta_{\underbrace{2\ldots20}_{i}} \cap frE_s] \cup \left[\bigcup_{i=1}^{\infty} \Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i}} \cap frE_s\right],$$

and the distances satisfy

$$\rho(\Delta_{\underbrace{2\ldots20}_{i}}, \Delta_{\underbrace{2\ldots20}_{i+1}}) = \rho(\Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i}}, \Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i+1}}) =$$

$$= \frac{s-2}{s^i(s-1)} > 0,$$

while frE_s is similar to $\Delta_{\underbrace{2\ldots20}_{i}} \cap frE_s = \Delta_{\underbrace{[s-1]\ldots[s-1][s+1]}_{i}} \cap frE_s$ with similarity coefficient $k_i = \frac{1}{s^i}$.

Therefore, its Hausdorff–Besicovitch dimension is equal to its N-similarity dimension, which is the solution of the equation

$$2\sum_{i=1}^{\infty} k_i^x = 1 \Longleftrightarrow 2\sum_{i=1}^{\infty} s^{-ix} = 1,$$

namely, $\dim_H frE_s = \log_s 3$.

Corollary 7. *The boundary of the Guthrie–Nymann Cantorval (the case $s = 4$) has Hausdorff–Besicovitch dimension equal to* $\log_4 3$.

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