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Fibonacci and Lucas numbers close to Narayana's cows numbers

Abstract. In this study, we find all Fibonacci and Lucas numbers which are close to Narayana's cows numbers. The main tools used in this work are lower bounds for linear forms in logarithms due to Matveev and Dujella-Pethö version of the Baker-Davenport reduction method in Diophantine approximation.
Key words: Fibonacci number, Lucas number, Narayana's cows number, Linear forms in logarithms, Reduction method

Анотація. У цьому дослідженні ми знаходимо всі числа Фібоначчі та Люка, які є близькими до чисел корів Нараяни. Основними інструментами, що використовуються в цій роботі, є оцінки знизу для лінійних форм від логарифмів, отримані Матвєєвим, а також версія Дуєлли–Петьо методу редукції Бейкера–Давенпорта у діофантових наближеннях
Ключові слова: числа Фібоначчі, числа Люка, числа корів Нараяни, лінійні форми від логарифмів, метод редукції

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$ with the initials $F_0 = 0$ and $F_1 = 1$ and let $\{L_n\}_{n\geq 0}$ be the Lucas sequence defined by $L_{n+2} = L_{n+1} + L_n$ where initials are $L_0 = 2$ and $L_1 = 1$. The characteristic equation for both of these sequences is $x^2 - x - 1 = 0$ having roots

$$(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

With these values of α and β, the Binet formulas for the sequences of Fibonacci and Lucas numbers are respectively given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 1. \tag{1.1}$$

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Using induction, it can be seen that the inequalities

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \qquad \text{and} \qquad \alpha^{n-1} \leq L_n \leq 2\alpha^n \quad \text{hold for all } n \geq 1. \tag{1.2}$$

The Narayana's cows sequence $\{N_m\}_{m\geq 0}$ is a ternary recurrent sequence given by

$$N_{m+3} = N_{m+2} + N_m \quad \text{for } m \geq 0,$$

with initials $N_0 = N_1 = N_2 = 1$. It is the sequence A000930 in the OEIS. The first few terms of the Narayana's cows sequence are

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots$$

Its characteristic polynomial is

$$f(x) = x^3 - x^2 - 1,$$

which is irreducible over $\mathbb{Q}[x]$ and has roots γ, δ and λ given by

$$\gamma = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$

$$\delta = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$

$$\lambda = \bar{\delta} = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3},$$

where $w = \frac{-1+i\sqrt{3}}{2}$. The Binet formula for the Narayana's cows sequence is given by

$$N_m = p\gamma^m + q\delta^m + r\lambda^m \quad \text{for all} \quad m \geq 0, \tag{1.3}$$

where

$$p = \frac{\gamma}{(\gamma-\delta)(\gamma-\lambda)}, \quad q = \frac{\delta}{(\delta-\gamma)(\delta-\lambda)}, \quad r = \frac{\lambda}{(\lambda-\gamma)(\lambda-\delta)}.$$

The formula (1.3) can also be written in the form

$$N_m = C_\gamma\gamma^{m+2} + C_\delta\delta^{m+2} + C_\lambda\lambda^{m+2} \quad \text{for all} \quad m \geq 0, \tag{1.4}$$

where

$$C_x = \frac{1}{x^3+2}, \quad x \in \{\gamma, \delta, \lambda\}.$$

The coefficient C_γ has the minimal polynomial $31x^3 - 31x^2 + 10x - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle. One can approximate the following:

$$\gamma \approx 1.46557;\ |\delta| = |\lambda| \approx 0.826031;\ |C_\delta\delta^{m+2} + C_\lambda\lambda^{m+2}| < 1/2 \quad \text{for all} \quad m \geq 1.$$

For every positive integer $m \geq 1$, we have

$$\gamma^{m-2} \leq N_m \leq \gamma^{m-1}. \tag{1.5}$$

An integer n is said to be close to a positive integer m if it satisfies

$$|n - m| < \sqrt{m}.$$

By using the above concept, Chern and Cui [7] found all the Fibonacci numbers which are close to a power of 2. Later, Bravo et al. [5] extended the previous work of Chern and Cui [7] by taking into consideration the k-generalized Fibonacci sequence where they found all k-Fibonacci numbers which are close to a power of 2. Following that, Hasanlizade [9] extended the previous work of [7] by considering the sum of Fibonacci numbers and studied the sum of two Fibonacci numbers close to a power of 2. In [12], Tripathy and Patel searched for the sum of three Fibonacci numbers which are close to a power of 2. Recently, Açikel et al. [1] determined the k-generalized Lucas numbers which are close to a power of 2. Consequently, Bachabi and Togbé [2] studied k-Pell numbers which are close to power of 2. Most recently, Pomeo and Bravo [10] determined all members of the Pell sequence which are close to Fibonacci numbers.

Motivated by the aforementioned studies, the present work introduces a systematic and unified investigation into Diophantine equations involving the classical second order sequences of Fibonacci and Lucas numbers together with the third order Narayana's cows sequence. By examining the notion of "closeness" between terms of sequences of distinct orders, we gain insight into how growth and distributional behaviors differ among linear recurrence families. Consequently, the Narayana sequence emerges as a natural higher order analogue for such a comparative analysis. Specifically, we consider the Diophantine inequalities

$$|F_n - N_m| < N_m^{1/2} \tag{1.6}$$

and

$$|L_n - N_m| < N_m^{1/2} \tag{1.7}$$

in nonnegative integers n and m. Our results reveal structural connections between these sequences and provide a methodological framework that opens avenues for similar studies involving higher order recurrence sequences such as the Padovan, Perrin and Tribonacci numbers.

To be more precise, we prove the following theorems.

Theorem 1. *The only solutions of the Diophantine inequality* (1.6) *in nonnegative integers n and m are given by*

$$(n, m) \in \{(1,0), (1,1), (1,2), (1,3), (2,0), (2,1), (2,2), (2,3), (3,3), (3,4), (4,3),$$

$$(4,4), (4,5), (5,5), (5,6), (6,6), (6,7), (7,8), (8,9), (10,12), (11,13), (15,18)\}.$$

Theorem 2. *The only solutions of the Diophantine inequality* (1.7) *in nonnegative integers* n *and* m *are given by*

$$(n,m) \in \{(0,3),(0,4),(1,0),(1,1),(1,2),(1,3),(2,3),(2,4),(2,5),(3,4),(3,5),$$
$$(3,6),(4,6),(4,7),(5,7),(5,8),(6,9),(7,10),(8,11),(10,14),(11,15),(14,19)\}.$$

To prove our main results, we first rewrite equation (1.6) and (1.7) into a appropriate linear form in logarithms of algebraic numbers. We then apply Matveev's lower bound to obtain absolute upper bounds for m and n, followed by the Dujella and Pethö [8] reduction method to bring these bounds within a manageable range. Computational verification for small parameters is carried out using *Mathematica*. The proofs rely on preliminary results discussed in the next section

2. Preliminary results

This section is devoted to gathering several definitions, notations, properties, and results that will be used in the rest of this study.

2.1. Linear forms in logarithms

Let γ be an algebraic number of degree d with a minimal primitive polynomial

$$f(Y) := b_0 Y^d + b_1 Y^{d-1} + \cdots + b_d = b_0 \prod_{j=1}^{d} (Y - \gamma^{(j)}) \in \mathbb{Z}[Y],$$

where the b_j's are relatively prime integers, $b_0 > 0$, and the $\gamma^{(j)}$'s are conjugates of γ. Then the *logarithmic height* of γ is given by

$$h(\gamma) = \frac{1}{d}\left(\log b_0 + \sum_{j=1}^{d} \log\left(\max\{|\gamma^{(j)}|, 1\}\right)\right).$$

With the above notation, Matveev (see [6, Theorem 9.4]) proved the following result.

Theorem 3. *Let* $\eta_1, \ldots, \eta_s$ *be positive real algebraic numbers in a real algebraic number field* $\mathbb{L}$ *of degree* $d_{\mathbb{L}}$. *Let* $a_1, \ldots, a_s$ *be non-zero integers such that*

$$\Lambda := \eta_1^{a_1} \cdots \eta_s^{a_s} - 1 \neq 0.$$

Then

$$-\log|\Lambda| \leq 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) \cdot B_1 \cdots B_s,$$

where

$$D \geq \max\{|a_1|, \ldots, |a_s|\},$$

and

$$B_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \text{ for all } j = 1, \ldots, s.$$

2.2. The reduction method

Our next tool is a version of the reduction method of Baker and Davenport (see [3]). Here, we use a slight variant of the version given by Dujella and Pethö (see [8]). For a real number x, we write $||x||$ for the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\epsilon := ||\mu q|| - M \cdot ||\tau q||.$$

If $\epsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

3. The proof of Theorem 1

Let us consider (n, m) to be a nonnegative integer solution to the Diophantine inequality (1.6). Consider two different cases in m.
Case $m \leq 250$. In this case, combining (1.2) and (1.6) with the fact F_n is close to N_m yields

$$\alpha^{n-2} \leq F_n < N_{250} + N_{250}^{1/2} < 1.95 \times 10^{41},$$

which implies $n \leq 199$. We search for the solutions of (1.6) using *Mathematica* in the range $m \leq 250$ and $n \leq 199$, which are listed in Theorem 1. So, for the rest of the proof, we consider $m > 250$.
Case $m > 250$. Here, since the function $x \mapsto x - x^{1/2}$ is increasing for $x > 1/4$, it can be seen that

$$2.85 \times 10^{41} < N_{251} - N_{251}^{1/2} \leq N_m - N_m^{1/2} < F_n \leq \alpha^{n-1}.$$

So, it must be that $n \geq 199$.

Lemma 2. *If integers m and n with $m > 250$ satisfy the Diophantine equation* (1.6)*, then we have the following inequalities*

$$\frac{7m}{10} < n < \frac{4m}{5}. \tag{3.1}$$

Proof. Combining the estimates (1.6) and (1.5) with the equation (1.2), we have

$$\alpha^{n-2} \leq F_n < \gamma^{(m-1)/2} + \gamma^{m-1} < 1.1\gamma^{m-1} \tag{3.2}$$

and

$$0.99\gamma^{m-2} < \gamma^{m-2} - \gamma^{(m-1)/2} \leq N_m - N_m^{1/2} < F_n \leq \alpha^{n-1} \tag{3.3}$$

which hold for all $m > 250$. Taking logarithms on both sides of inequalities (3.2) and (3.3) and combining them, we obtain

$$n < 2 + \frac{\log 1.1}{\log \alpha} + (m-1)\frac{\log \gamma}{\log \alpha} < \frac{4m}{5} \text{ and } m < 1 + (m-2)\frac{\log \gamma}{\log \alpha} + \frac{\log 0.99}{\log \alpha} < \frac{10n}{7}.$$

This leads to the inequalities

$$\frac{7m}{10} < n < \frac{4m}{5}.$$

This finishes the proof.

Next, we get the following result, which gives an upper bound of n and m with the help of Lemma 2.

Lemma 3. *If (m, n) is a solution of the Diophantine equation (1.6) with $m > 250$ and $n \geq 201$, then m and n are bounded as*

$$m < 8.12 \times 10^{16} \quad \textit{and} \quad n < 6.5 \times 10^{16}.$$

Proof. Using Binet's formula (1.1) and (1.4), we write the equation (1.6) as

$$\left| \frac{\alpha^n}{\sqrt{5}} - C_\gamma \gamma^{m+2} \right| < \gamma^{(m-1)/2} + \frac{|\beta|^n}{\sqrt{5}} + \left| C_\delta \delta^{m+2} + C_\lambda \lambda^{m+2} \right| < \gamma^{(m-1)/2} + 1 < \gamma^{m/2}$$

holds for $m > 250$ and $n \geq 201$. Dividing across by $C_\gamma \gamma^{m+2}$ gives us

$$|\Lambda_1| \leq \left| \frac{\alpha^n \gamma^{-(m+2)}}{\sqrt{5} C_\gamma} - 1 \right| < \frac{2.4}{\gamma^{m/2}}, \tag{3.4}$$

where

$$\Lambda_1 := \alpha^n \gamma^{-(m+2)} (\sqrt{5} C_\gamma)^{-1} - 1. \tag{3.5}$$

To apply Theorem 3, we need to show that $\Lambda_1 \neq 0$. Indeed, $\Lambda_1 = 0$ implies

$$\frac{\alpha^n}{\sqrt{5}} = C_\gamma \gamma^{m+2}. \tag{3.6}$$

Let G be the Galois group of $Q(\alpha, \gamma)$ over Q. Conjugating the above relation (3.6) by the automorphism $\sigma \in G$ such that $\sigma(\alpha) = \beta$ and $\sigma(\gamma) = \gamma$, we have

$$1 < C_\gamma \gamma^{m+2} = \frac{|\beta|^n}{\sqrt{5}}.$$

But this is impossible as $|\beta| < 1$. Thus, $\Lambda_1 \neq 0$. Therefore, we apply Theorem 3 to get a lower bound for Λ_1 given by (3.5) with the parameters:

$$\eta_1 := \alpha, \quad \eta_2 := \gamma, \quad \eta_3 := \sqrt{5}C_\gamma,$$

and

$$a_1 := n, \quad a_2 := -(m+2), \quad a_3 := -1.$$

Note that the algebraic numbers η_1, η_2, η_3 belongs to the field $\mathbb{L} := \mathbb{Q}(\alpha, \gamma)$, so we can assume $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 6$. Since $h(\eta_1) = (\log \alpha)/2$ and $h(\eta_2) = (\log \gamma)/3$, it follows that

$$\max\{6h(\eta_1), |\log \eta_1|, 0.16\} = 3\log \alpha := B_1.$$

and

$$\max\{6h(\eta_2), |\log \eta_2|, 0.16\} = 2\log \gamma := B_2.$$

$$h(\eta_3) \leq h(\sqrt{5}) + h(C\gamma) < \frac{\log 5}{2} + \frac{\log 31}{3} = 1.95.$$

Thus, we obtain

$$\max\{6h(\eta_3), |\log \eta_3|, 0.16\} = 11.7 := B_3.$$

In addition, by (3.1) $\max\{|n|, |-(m+2)|, |-1|\} = m+2$. So we can take $D := m+2$. Applying Matveev's theorem we get

$$|\Lambda_1| > \exp\big(-1.4 \times 30^6 \times 3^{4.5} \times (6)^2(1+\log 6)(1+\log(m+2))(3\log \alpha)(2\log \gamma)(11.7)\big). \tag{3.7}$$

From the comparison of lower bound (3.7) and upper bound (3.4) of $|\Lambda_1|$ gives us

$$\frac{m}{2}\log \gamma - \log 2.4 < 2.23 \times 10^{14} \log m \tag{3.8}$$

where we used the fact that inequality $1 + \log(m+2) < 1.2 \log m$ holds for $m > 250$. The last inequality (3.8), becomes

$$\frac{m}{\log m} < 1.17 \times 10^{15}. \tag{3.9}$$

We next use the fact that if $m \geq 1$, $S \geq (4m^2)^m$ and $\frac{x}{(\log x)^m} < S$, then $x < 2^m S(\log S)^m$. Thus, putting $S := 1.17 \times 10^{15}$ in (3.9) and performing the respective calculations, gives $m < 8.12 \times 10^{16}$. Therefore, by (3.1), we obtain $n < 6.5 \times 10^{16}$. This completes the proof of Lemma 3.

3.1. Reducing the bounds

We begin this subcase by formulating some facts concerning the exponential function.

Lemma 4. ([4], Lemma 8) *For any non-zero real number x, we have*
(a) $0 < x < |e^x - 1|$.
(b) If $x < 0$ *and* $|e^x - 1| < 1/2$, *then* $|x| < 2|e^x - 1|$.

The upper bound on m obtained in Lemma 3 is too large. Thus, we need to apply Lemma 1 to reduce it. Let us define

$$\Gamma_1 := n \log \alpha - (m+2) \log \gamma + \log(\sqrt{5} c_\gamma)^{-1}. \tag{3.10}$$

Then $e^{\Gamma_1} - 1 := \Lambda_1$, where Λ_1 is defined by (3.5). Therefore, (3.4) implies that

$$|e^{\Gamma_1} - 1| < \frac{2.4}{\gamma^{\frac{m}{2}}}. \tag{3.11}$$

Note that $\Gamma_1 \neq 0$. Thus, we distinguish the following cases. If $\Gamma_1 > 0$, then we can apply Lemma 4 (a) to obtain

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 < \frac{2.4}{\gamma^{\frac{m}{2}}}.$$

If $\Gamma_1 < 0$, then from (3.4) we have that $|e^{\Gamma_1} - 1| < 1/2$ and therefore $e^{|\Gamma_1|} < 2$. Thus, by Lemma 4 (b), we have that

$$0 < |\Gamma_1| \leq e^{|\Gamma_1|} - 1 = e^{|\Gamma_1|} |e^{\Gamma_1} - 1| < \frac{4.8}{\gamma^{\frac{m}{2}}}.$$

So in both cases, we have

$$0 < |\Gamma_1| < 4.8 \cdot \gamma^{-\frac{m}{2}}. \tag{3.12}$$

Inserting (3.10) in (3.12) and dividing both sides by $\log \gamma$, it results that

$$0 < \left| n \left(\frac{\log \alpha}{\log \gamma} \right) - (m+2) + \frac{\log(\sqrt{5} c_\gamma)^{-1}}{\log \gamma} \right| < 12.56 \cdot \gamma^{-\frac{m}{2}}. \tag{3.13}$$

In order to apply Lemma 1, we set

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := -2 + \frac{\log(\sqrt{5} c_\gamma)^{-1}}{\log \gamma}, \quad A := 12.56 \quad \text{and} \quad B := \gamma.$$

Therefore, the inequality (3.13), yields

$$0 < |n\tau - m + \mu| < A \cdot B^{-\frac{m}{2}}. \tag{3.14}$$

Note that $\tau = \frac{\log \alpha}{\log \gamma} \notin \mathbb{Q}$ because if $\frac{\log \alpha}{\log \gamma} = \frac{s}{t}$ for some coprime positive integers s and t, we would have $\gamma^s = \alpha^t \in \mathbb{Z}$. Using the automorphism σ defined above, we get $1 < \gamma^s = |\beta|^t < 1$, a contradiction. The fact $n < 6.5 \times 10^{16}$ by

Lemma 3 implies that we can take $M := 6.5 \times 10^{16}$. A quick computation with *Mathematica* reveals that the convergent

$$\frac{p_{40}}{q_{40}} = \frac{3511279069780293827}{2789144197090847260}$$

of τ is the first such that $q_{40} > 6M$ and $\epsilon > 0$. Therefore, if the inequality (3.14) has a solution, then

$$\frac{m}{2} < \frac{\log(Aq_{40}/\epsilon)}{\log B} < 120,$$

which implies that $m \leq 238$. However, this contradicts our assumption that $m > 250$. Thus, inequality (1.6) has no solutions with $m > 250$. This completes the proof of Theorem 1.

4. The proof of Theorem 2

Assume that the Diophantine inequality (1.7) has a non-negative integer solution for (m, n). Let us consider two cases on m.

Case $m \leq 280$. In this case, combining (1.2) and (1.7) with the fact that L_n is close to N_m gives

$$\alpha^{n-1} \leq L_n < N_{280} + N_{280}^{1/2} < 1.86 \times 10^{46},$$

which implies $n \leq 222$. Using *Mathematica*, we checked that all the solutions of the Diophantine inequality (1.7) in the range $m \leq 280$ and $n \leq 222$, which are listed in Theorem 2. Therefore, for the remainder of the paper, we assume that $m > 280$.

Case $m > 280$. Here, since the function $x \mapsto x - x^{1/2}$ is increasing for $x > 1/4$, it is easily seen that

$$2.72 \times 10^{46} < N_{281} - N^{1/2}{}_{281} \leq N_m - N^{1/2}{}_m < L_n \leq 2\alpha^n.$$

Hence, it must be $n \geq 220$.

Lemma 5. *If integers m and n with $m > 250$ satisfy the Diophantine equation* (1.7)*, then we have the following inequalities*

$$\frac{7m}{10} < n < \frac{4m}{5}. \tag{4.1}$$

Proof. Using the inequalities (1.2) and (1.5) and inequality (1.7), we obtain

$$\alpha^{n-1} < L_n < \gamma^{(m-1)/2} + \gamma^{m-1} < 1.1\gamma^{m-1} \tag{4.2}$$

and

$$0.99\gamma^{m-2} < \gamma^{m-2} - \gamma^{(m-1)/2} \leq N_m - N_m^{1/2} < L_n \leq 2\alpha^n \tag{4.3}$$

which hold for all $m > 280$. Combining the logarithms on both sides of inequalities (4.2) and (4.3), results in

$$n < 1 + \frac{\log 1.1}{\log \alpha} + (m-1)\frac{\log \gamma}{\log \alpha} < \frac{4m}{5}$$

and

$$m < \frac{\log 0.99}{\log \alpha} + (m-2)\frac{\log \gamma}{\log \alpha} - \frac{\log 2}{\log \alpha} < \frac{10n}{7}.$$

Therefore, we deduce that

$$\frac{7m}{10} < n < \frac{4m}{5}.$$

This completes the proof.

Next, we will find upper bounds of n and m with the help of Lemma 5. More precisely, we will show the following lemma.

Lemma 6. *If (m, n) is a solution of the Diophantine equation (1.7) with $m > 280$ and $n \geq 220$, then m and n are bounded as*

$$m < 4.68 \times 10^{16} \quad \textit{and} \quad n < 3.74 \times 10^{16}.$$

Proof. Using Binet's formula (1.1) and (1.4), the equation (1.7) can be rearranged as

$$\left|\alpha^n - C_\gamma \gamma^{m+2}\right| < \gamma^{(m-1)/2} + |\beta|^n + \left|C_\delta \delta^{m+2} + C_\lambda \lambda^{m+2}\right| < \gamma^{(m-1)/2} + 1.5 < \gamma^{m/2}$$

holds for $m > 280$ and $n \geq 220$. Dividing through by $C_\gamma \gamma^{m+2}$ yields

$$\left|\alpha^n \gamma^{-(m+2)} {C_\gamma}^{-1} - 1\right| < \frac{2.4}{\gamma^{m/2}}.$$

Let

$$\Lambda_2 := \alpha^n \gamma^{-(m+2)} {C_\gamma}^{-1} - 1. \tag{4.4}$$

From (4.4), we have

$$|\Lambda_2| < 2.4 \cdot \gamma^{-(m/2)}. \tag{4.5}$$

We have $\Lambda_2 \neq 0$, otherwise we would get

$$\alpha^n = C_\gamma \gamma^{m+2}. \tag{4.6}$$

Conjugating the above relation (4.6) by some automorphism $\sigma \in G$ such that $\sigma(\alpha) = \beta$ and $\sigma(\gamma) = \gamma$ of the Galois group G of the decomposition field of $Q(\alpha, \gamma)$ over Q, we have

$$1 < C_\gamma \gamma^{m+2} = |\beta|^n.$$

which leads to a contradiction as $|\beta| < 1$. Let us apply Theorem 3 to Λ_2 given by (4.4). To this end, we take as parameters:

$$(\eta_1, a_1) := (\alpha, n), \qquad (\eta_2, a_2) := (\gamma, -(m+2)), \quad \text{and} \quad (\eta_3, a_3) := (C_\gamma, -1).$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\alpha, \gamma)$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 6$. Since $h(\eta_1) = (\log \alpha)/2$ and $h(\eta_2) = (\log \gamma)/3$, it follows that

$$\max\{6h(\eta_1), |\log \eta_1|, 0.16\} = 3 \log \alpha := B_1$$

and

$$\max\{6h(\eta_2), |\log \eta_2|, 0.16\} = 2 \log \gamma := B_2.$$

$$h(\eta_3) \leq h(C\gamma) = \frac{\log 31}{3} = 1.14.$$

Thus, we obtain

$$\max\{6h(\eta_3), |\log \eta_3|, 0.16\} = 6.84 := B_3.$$

Finally, from Lemma 5 we can choose $D := m+2 = max\{|n|, |-(m+2)|, |-1|\}$. Therefore, according to Theorem 3, we have

$$|\Lambda_2| > \exp\big(- 1.4 \times 30^6 \times 3^{4.5} \times (6)^2 (1 + \log 6)(1 + \log(m+2))(6.84)(3 \log \alpha) (2 \log \gamma)\big). \tag{4.7}$$

Comparing the obtained bounds of (4.7) and (4.5) of $|\Lambda_2|$ gives us

$$\frac{m}{2} \log \gamma - \log 2.4 < 1.3 \times 10^{14} \log m \tag{4.8}$$

where we used the fact that inequality $1 + \log(m+2) < 1.2 \log m$ holds for $m > 280$. So, the last inequality (4.8), becomes

$$\frac{m}{\log m} < 6.85 \times 10^{14}. \tag{4.9}$$

We next use the fact that if $m \geq 1$, $S \geq (4m^2)^m$ and $\frac{x}{(\log x)^m} < S$, then $x < 2^m S (\log S)^m$. Thus putting $S := 6.85 \times 10^{14}$ in (4.9) and performing the respective calculations, yields $m < 4.68 \times 10^{16}$. Therefore, by (4.1), we obtain $n < 3.74 \times 10^{16}$. This completes the proof of Lemma 6.

4.1. Reducing the bounds

We use the Lemma 1 to reduce the bound on m. To do so, we first set

$$\Gamma_2 := n \log \alpha - (m+2) \log \gamma + \log(C_\gamma^{-1}). \tag{4.10}$$

Therefore, inequality (4.5) can be rewritten as $|e^{\Gamma_2} - 1| < 2.4/\gamma^{m/2}$. In this case, $\Gamma_2 \neq 0$. If $\Gamma_2 > 0$, then we can apply Lemma 4 (a) to obtain $|\Gamma_2| < 2.4/\gamma^{m/2}$. Now, if $\Gamma_2 < 0$, then $|e^{\Gamma_2} - 1| < 1/2$. Thus, by Lemma 4 (b), we have that $|\Gamma_2| < 2|e^{\Gamma_2} - 1| < 4.8/\gamma^{m/2}$. In any case, we get that

$$|\Gamma_2| < 4.8 \cdot \gamma^{-\frac{m}{2}}. \tag{4.11}$$

Putting (4.10) in (4.11) and dividing both sides by $\log \gamma$, we get that

$$0 < |n\tau - m + \mu| < A \cdot B^{-\frac{m}{2}} \tag{4.12}$$

where

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := -2 + \frac{\log\left(C_\gamma^{-1}\right)}{\log \gamma}, \quad A := 12.56 \quad \text{and} \quad B := \gamma.$$

Now, we apply Lemma 1 to (4.12) by taking $M := 3.74 \times 10^{16}$ which is an upper bound on n by Lemma 6. With the help of *Mathematica*, we found that the denominator of the 38th convergent

$$\frac{p_{38}}{q_{38}} = \frac{936188796148115612}{743650816772409797}$$

of τ is the first such that $q_{38} > 6M$ and $\epsilon > 0$. Furthermore, if the inequality (4.12) has a solution, then

$$\frac{m}{2} < \frac{\log(Aq_{38}/\epsilon)}{\log B} < 120,$$

which implies that $m \leq 238$. However, this contradicts our assumption that $m > 280$. Thus, inequality (1.7) has no solutions with $m > 280$. This completes the proof of Theorem 2.

Concluding remarks. This paper determines all Fibonacci and Lucas numbers that are close to Narayana's cows numbers, using methods from Diophantine approximation such as linear forms in logarithms and reduction techniques. The results reveal a finite set of exact coincidences between these second order and third order sequences, deepening the understanding of their structural relationships. The approach presented here provides a framework that can be extended to study similar Diophantine problems involving other higher order recurrence sequences, such as the Padovan, Perrin, and Tribonacci sequences, as well as their generalizations.

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