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Submonoids of integer group determinants

Abstract. For a finite group G, let $S(G)$ denote the set of all integer values obtained from the group determinant by substituting integers for its variables. Such values are called integer group determinants. Giving a complete description of $S(G)$ is a difficult problem, known as the Taussky–Todd integer group determinant problem.
In this paper, we introduce a submonoid $S'(G)$ of the monoid $S(G)$ arising from a specific specialization of the group determinant, and provide a framework for investigating certain aspects of the lower structure of $S(G)$. For finite groups G of order at most 5, we determine $S'(G)$ explicitly and present concrete examples of integer group determinants.
The motivation for considering the submonoid $S'(G)$ is that specialization patterns can retain meaningful arithmetic information. In particular, this viewpoint may be useful in contexts related to classical number-theoretic problems, such as the search for Wieferich primes.

Key words: Group determinant, Integer group determinant, Monoid

Анотація. Нехай для скінченної групи G символ $S(G)$ позначає множину всіх цілих значень, які набуває груповий визначник при підстановці цілих чисел замість його змінних. Такі значення називають цілочисельними груповими визначниками. Повний опис множини $S(G)$ є складною задачею, відомою як проблема Таусскі–Тодда про цілочисельні групові визначники.
У цій статті ми вводимо підмоноїд $S'(G)$ моноїда $S(G)$, що виникає внаслідок певної спеціалізації групового визначника, і пропонуємо підхід до дослідження деяких аспектів нижньої структури $S(G)$. Для скінченних груп G порядку щонайбільше 5 ми явно знаходимо $S'(G)$ та наводимо конкретні приклади цілочисельних групових визначників.
Мотивацією для розгляду підмоноїда $S'(G)$ є те, що схеми спеціалізації здатні зберігати змістовну арифметичну інформацію. Зокрема, такий підхід може бути корисним у контексті класичних задач теорії чисел, таких як пошук простих чисел Віферіха.

Ключові слова: груповий визначник, цілочисельний груповий визначник, моноїд

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1. Introduction

The group determinant, originally introduced by Dedekind [15] and subsequently developed by Frobenius [3], is a classical object in the representation theory of finite groups. Let G be a finite group and let x_g be a variable associated with each element $g \in G$. We denote by $\mathbb{Z}[x_g]$ the multivariate polynomial ring in the variables x_g over $\mathbb{Z}$. The group matrix $M_G(x_g)$ and the group determinant $\Theta_G(x_g)$ are defined as follows:

$$M_G(x_g) := (x_{gh^{-1}})_{g,h\in G}, \qquad \Theta_G(x_g) := \det M_G(x_g) \in \mathbb{Z}[x_g].$$

Frobenius showed that $\Theta_G(x_g)$ admits a factorization in terms of the irreducible representations of G, a result that played a fundamental role in the development of character theory.

When integers are substituted for the variables x_g, the resulting values of $\Theta_G(x_g)$ are called integer group determinants. We define

$$S(G) := \{\Theta_G(a_g) \mid a_g \in \mathbb{Z}\}.$$

It is easy to see that $S(G)$ forms a commutative monoid under multiplication. The problem of giving a complete description of $S(G)$ was posed by Olga Taussky–Todd [14], and is now known as the Taussky–Todd integer group determinant problem.

Integer group determinants are closely related to classical problems in number theory. A prime number p is called a Wieferich prime if it satisfies

$$2^{p-1} \equiv 1 \pmod{p^2}.$$

As of this writing, only two Wieferich primes are known.

A remarkable connection between Wieferich-type congruences and integer group determinants was established in [20]: if p is a prime and G is a finite abelian group of order p^n with a cyclic direct factor of order p, then for any prime q,

$$q \in S(G) \iff q^{p-1} \equiv 1 \pmod{p^n}.$$

In particular, determining whether $2 \in S(\mathrm{C}_p^2)$ is equivalent to asking whether p is a Wieferich prime, where C_p denotes the cyclic group of order p. Historically, Wieferich primes arise as a necessary condition for the first case of Fermat's Last Theorem [16].

Consequently, obtaining a complete description of $S(G)$ would, in principle, make it possible to decide the Wieferich property of primes through group determinants. However, such a complete description is known only for restricted classes of finite groups. For example, it is known for the cyclic groups C_p and C_{2p} [5,8], for the dihedral groups D_{2p} and D_{4p} with p an odd prime [2,6], and for all groups of order at most 16 [1, 2, 4, 7, 9, 11–13, 17–19, 21–24]. Moreover, Paudel and Pinner [10] determined the integer group determinants for the

remaining four groups of order 18, thereby completing the resolution of the Taussky–Todd integer group determinant problem for all finite groups of order less than 20. Despite this substantial progress, a complete description of $S(G)$ remains unknown in general.

Lower bounds for $S(G)$ have also been studied [17]. If H is an abelian subgroup of a finite group G, then

$$\{a^{[G:H]} \mid a \in S(H)\} \subset S(G) \subset S(H),$$

where $[G : H]$ denotes the index of H in G. This provides a general lower bound for $S(G)$, but it does not lead to a complete description.

Because a complete determination of the monoid $S(G)$ is often out of reach, it is natural to focus on submonoids that admit complete descriptions. Motivated by this perspective, we introduce a submonoid $S'(G)$, defined through a structured specialization of the group determinant.

Let T denote the $|G| \times |G|$ matrix whose entries are all equal to 1. We define the block matrix

$$M'_G(x_g; x) := \begin{pmatrix} M_G(x_g) & xT \\ xT & M_G(x_g) \end{pmatrix},$$

and its determinant

$$\Theta'_G(x_g; x) := \det M'_G(x_g; x).$$

We then define

$$S'(G) := \{\Theta'_G(a_g; a) \mid a_g, a \in \mathbb{Z}\}.$$

We emphasize that $S'(G)$ depends on a specific but structured specialization pattern of the group determinant. By construction, $S'(G)$ is a commutative monoid under multiplication.

The main results of this paper are given in the following two theorems.

Theorem 1. *Let G be a finite group. Then the following hold:*

(1) $S'(G) \subset S(G)$.

(2) *Let H be a subgroup of G. If $[G : H] = 2$, then*

$$\{a^2 \mid a \in S(H)\} \subset S'(H) \subset S(G).$$

The next theorem gives a complete description of the monoid $S'(G)$ for all finite groups of order at most 5.

Theorem 2. *For every finite group G of order at most 5, the monoid $S'(G)$ is given as follows:*

(1) $S'(\mathrm{C}_1) = \{2k + 1,\ 2^2k \mid k \in \mathbb{Z}\} = S(\mathrm{C}_2)$.

(2) $S'(\mathrm{C}_2) = \{4k + 1,\ 2^4(2k + 1),\ 2^6k \mid k \in \mathbb{Z}\} = S(\mathrm{C}_2^2)$.

(3) $S'(\mathrm{C}_3) = \{6k+1,\ 2^2(3k+1),\ 3^4(2k+1),\ 2^2 3^4 k \mid k \in \mathbb{Z}\}$.

(4) $S'(\mathrm{C}_4) = \{8k+1,\ 2^8(2k+1),\ 2^{10}k \mid k \in \mathbb{Z}\}$.

(5) $S'(\mathrm{C}_2^2) = \{8k+1,\ 2^8(4k+1),\ 2^{12}(2k+1),\ 2^{14}k \mid k \in \mathbb{Z}\}$.

(6) $S'(\mathrm{C}_5) = \{k(k+10l) \mid k \in \mathbb{Z}_5^*, l \in \mathbb{Z}\} \cup \{5^4(2k+1), 2^2 5^4 k \mid k \in \mathbb{Z}\}$.

Here, $\mathbb{Z}_5^*$ denotes the set of all integers relatively prime to 5.

The main results of this paper provide a complete description of the submonoid $S'(G)$ for all finite groups G of order at most 5. In particular, we show that $S'(G)$ is closely related to $S(G)$ while admitting a complete description in these cases.

This suggests that structured specialization patterns of the group determinant can preserve nontrivial arithmetic information. Although this approach does not immediately yield new Wieferich primes, it offers a framework for extracting and isolating arithmetic features of integer group determinants through more accessible submonoids.

This paper is organized as follows. In Section 2, we prove Theorem 1, which establishes the basic structural properties of the submonoid $S'(G)$ and its relationship with $S(G)$. Section 3 is devoted to the proof of Theorem 2 (1)–(5). In Section 4, we prove Theorem 2 (6).

2. Proof of Theorem 1

To prove Theorem 1, we use the following lemma:

Lemma 1. *Let G be a finite group. Then we have*

$$\Theta'_G(x_g; x) = \Theta_G(x_g + x)\Theta_G(x_g - x).$$

Proof. Since the matrices $M_G(x_g)$ and xT commute, we have

$$\begin{aligned}\Theta'_G(x_g; x) &= \det\left(M_G(x_g)^2 - x^2T^2\right)\\ &= \det\left(M_G(x_g) + xT\right)\det\left(M_G(x_g) - xT\right)\\ &= \Theta_G(x_g + x)\Theta_G(x_g - x).\end{aligned}$$

This completes the proof.

Proof of Theorem 1. From Lemma 1 and the fact that $S(G)$ is a monoid, (1) holds. We prove (2). Let $G = Ht_1 \sqcup Ht_2$, let $H = \{h_1, h_2, \ldots, h_m\}$ and let

$$g_1 := h_1 t_1,\ \ldots,\ g_m := h_m t_1,\ g_{m+1} := h_1 t_2,\ \ldots,\ g_{2m} := h_m t_2.$$

Then we have

$$M_G(x_g) = (x_{g_i g_j^{-1}})_{1\le i,j\le 2m} = \begin{pmatrix} M_H(x_h) & (x_{h_i t_1 t_2^{-1} h_j^{-1}})_{1\le i,j\le m} \\ (x_{h_i t_2 t_1^{-1} h_j^{-1}})_{1\le i,j\le m} & M_H(x_h)\end{pmatrix}.$$

For any $g \in G \setminus H$, let $x_g = x$. Then $M_G(x_g) = M'_H(x_h; x)$. Therefore, $S'(H) \subset S(G)$. Also, from Lemma 1, we have $\Theta_H(x_h)^2 = \Theta'_H(x_h; 0)$. Therefore,

$$\{a^2 \mid a \in S(H)\} \subset S'(H).$$

This completes the proof.

3. Proof of Theorem 2 (1)–(5)

To prove Theorem 2 (1)–(5), we use the following theorem, lemmas and corollary:

Theorem 3 ([13]). *The following hold:*

(1) $S(\mathrm{C}_2) = \{2k+1,\ 2^2k \mid k \in \mathbb{Z}\}$.

(2) $S(\mathrm{C}_2^2) = \{4k+1,\ 2^4(2k+1),\ 2^6k \mid k \in \mathbb{Z}\}$.

(3) $S(\mathrm{C}_6) = \{6k \pm 1,\ 2^2(3k+1),\ 2^2(3k+2),\ 3^2(2k+1),\ 2^2 3^2 k \mid k \in \mathbb{Z}\}$.

(4) $S(\mathrm{C}_4 \times \mathrm{C}_2) = \{8k+1,\ 2^8k \mid k \in \mathbb{Z}\}$.

(5) $S(\mathrm{C}_2^3) = \{8k+1,\ 2^8(4k+1),\ 2^{12}k \mid k \in \mathbb{Z}\}$.

For any $\bar{i} \in \mathrm{C}_n$ with $i \in \{0, 1, \ldots, n-1\}$, we denote the variable $x_{\bar{i}}$ by x_i and the integer $a_{\bar{i}}$ by a_i.

Lemma 2. *Let $G = \mathrm{C}_3$. Then the following hold:*

(i) *If $a_0 + a_1 + a_2 \equiv 0 \pmod 3$, then $\Theta'_G(a_g; a) \in 3^4\mathbb{Z}$.*

(ii) *If $a_0 + a_1 + a_2 \not\equiv 0 \pmod 3$, then $\Theta'_G(a_g; a) \equiv 1 \pmod 3$.*

Proof. By direct calculation, we have

$$\begin{aligned}\Theta'_G(x_g; x) = &\left\{(x_0+x_1+x_2)^2 - 3(x_0x_1 + x_1x_2 + x_0x_2)\right\}^2 \\ &\times \left\{(x_0+x_1+x_2)^2 - 9x^2\right\}.\end{aligned}$$

This leads to the lemma.

Lemma 3. *Let $G = \mathrm{C}_4$. Then the following hold:*

(i) *If $a_0 + a_1 + a_2 + a_3 \equiv 0 \pmod 2$, then*

$$\Theta'_G(a_g; a) \in \{2^8(2k+1) \mid k \in \mathbb{Z}\} \cup 2^{10}\mathbb{Z}.$$

(ii) *If $a_0 + a_1 + a_2 + a_3 \equiv 1 \pmod 2$, then $\Theta'_G(a_g; a) \in \{2k+1 \mid k \in \mathbb{Z}\}$.*

Proof. By direct calculation, we have

$$\Theta'_G(x_g;x) = \left\{(x_0-x_2)^2+(x_1-x_3)^2\right\}^2 (x_0-x_1+x_2-x_3)^2 \\ \times (x_0+x_1+x_2+x_3+4x)(x_0+x_1+x_2+x_3-4x).$$

This leads to the lemma.

Lemma 4. *Let $G = \mathrm{C}_2^2$ and let $b_0 := b_{(0,0)}$, $b_1 := b_{(1,0)}$, $b_2 := b_{(0,1)}$ and $b_3 := b_{(1,1)} \in \mathbb{Z}$. Then the following hold:*

(i) *If $b_0+b_1+b_2+b_3 \equiv 0 \pmod 2$, then*

$$\Theta'_G(b_g;b) \in \{2^8(4k+1),\ 2^{12}(2k+1) \mid k \in \mathbb{Z}\} \cup 2^{14}\mathbb{Z}.$$

(ii) *If $b_0+b_1+b_2+b_3 \equiv 1 \pmod 2$, then $\Theta'_G(b_g;b) \in \{2k+1 \mid k \in \mathbb{Z}\}$.*

Proof. Let $x_0 := x_{(0,0)}$, $x_1 := x_{(1,0)}$, $x_2 := x_{(0,1)}$ and $x_3 := x_{(1,1)}$. By direct calculation, we have

$$\Theta'_G(x_g;x) = \left\{(x_0-x_2)^2-(x_1-x_3)^2\right\}^2 \left\{(x_0+x_2+2x)^2-(x_1+x_3+2x)^2\right\} \\ \times \left\{(x_0+x_2-2x)^2-(x_1+x_3-2x)^2\right\}.$$

This leads to the lemma.

From Theorems 1 (2) and 3 and Lemmas 2–4, we have the following corollary.

Corollary 1. *The following hold:*

(1) $S'(\mathrm{C}_1) \subset \{2k+1,\ 2^2k \mid k \in \mathbb{Z}\} = S(\mathrm{C}_2)$.

(2) $S'(\mathrm{C}_2) \subset \{4k+1,\ 2^4(2k+1),\ 2^6k \mid k \in \mathbb{Z}\} = S(\mathrm{C}_2^2)$.

(3) $S'(\mathrm{C}_3) \subset \{6k+1,\ 2^2(3k+1),\ 3^4(2k+1),\ 2^2 3^4 k \mid k \in \mathbb{Z}\}$.

(4) $S'(\mathrm{C}_4) \subset \{8k+1,\ 2^8(2k+1),\ 2^{10}k \mid k \in \mathbb{Z}\}$.

(5) $S'(\mathrm{C}_2^2) \subset \{8k+1,\ 2^8(4k+1),\ 2^{12}(2k+1),\ 2^{14}k \mid k \in \mathbb{Z}\}$.

Lemma 5. *For any $k \in \mathbb{Z}$, the following are elements of $S'(\mathrm{C}_n)$:*

(1) $2nk+1$.

(2) $2^{2n}(nk+1)$.

(3) $2^2 n^4 k$.

(4) *If n is odd, then* $2^2(nk+1)$.

(5) $\dfrac{1}{3}(2nk+3)\left\{2^n+(-1)^{n+1}\right\}^2$.

To prove Lemma 5, we use the following theorem: *For a finite abelian group G, we have*

$$\Theta_G(x_g) = \prod_{\chi\in\widehat{G}}\sum_{g\in G}\chi(g)x_g,$$

where $\widehat{G}$ is the dual group of G. This is called Dedekind's theorem.

Proof of Lemma 5. First, we prove (1). Let $a_0 = k+1$, $a_g = k$ for $g \in \mathrm{C}_n \setminus \{\overline{0}\}$ and $a = k$. Then

$$\begin{aligned}\Theta_{\mathrm{C}_n}(a_g + a) &= \prod_{i=0}^{n-1}\sum_{j=0}^{n-1}\zeta_n^{ij}(a_j + a)\\ &= (2nk+1)\prod_{i=1}^{n-1}\left(1 + 2k\sum_{j=0}^{n-1}\zeta_n^{ij}\right)\\ &= 2nk+1,\\ \Theta_{\mathrm{C}_n}(a_g - a) &= \prod_{i=0}^{n-1}\sum_{j=0}^{n-1}\zeta_n^{ij}(a_j - a) = 1.\end{aligned}$$

Therefore, from Lemma 1, we obtain (1). We prove (2). Let $a_0 = k+2$, $a_g = k$ for $g \in \mathrm{C}_n \setminus \{\overline{0}\}$ and $a = k$. Then

$$\Theta_{\mathrm{C}_n}(a_g + a) = (2nk+2)\prod_{i=1}^{n-1}2 = 2^n(nk+1), \quad \Theta_{\mathrm{C}_n}(a_g - a) = 2^n.$$

Therefore, from Lemma 1, we obtain (2). We prove (3). Let $a_0 = k+2$, $a_1 = k$, $a_g = k+1$ for $g \in \mathrm{C}_n \setminus \{\overline{0}, \overline{1}\}$ and $a = k-1$. Then

$$\Theta_{\mathrm{C}_n}(a_g + a) = 2nk\prod_{i=1}^{n-1}(1-\zeta_n^i) = 2n^2k, \ \ \Theta_{\mathrm{C}_n}(a_g - a) = 2n\prod_{i=1}^{n-1}(1-\zeta_n^i) = 2n^2.$$

Therefore, from Lemma 1, we obtain (3). We prove (4). Let $a_0 = a_1 = k+1$, $a_g = k$ for $g \in \mathrm{C}_n \setminus \{\overline{0}, \overline{1}\}$ and $a = k$. Then

$$\Theta_{\mathrm{C}_n}(a_g + a) = (2nk+2)\prod_{i=1}^{n-1}(1+\zeta_n^i) = \begin{cases}0, & n \text{ is even;}\\ 2nk+2, & n \text{ is odd,}\end{cases}$$

$$\Theta_{\mathrm{C}_n}(a_g - a) = 2\prod_{i=1}^{n-1}(1+\zeta_n^i) = \begin{cases}0, & n \text{ is even;}\\ 2, & n \text{ is odd.}\end{cases}$$

Therefore, from Lemma 1, we obtain (4). We prove (5). Let $a_0 = -k-1$, $a_g = -k$ for $g \in \mathrm{C}_n \setminus \{\overline{0}, \overline{n-1}\}$, $a_{n-1} = -k-2$ and $a = -k$. Then

$$\begin{aligned}\Theta_{\mathrm{C}_n}(a_g + a) &= -(2nk+3)2^{n-1}\prod_{i=1}^{n-1}\left(-\frac{1}{2} - \zeta_n^{i(n-1)}\right) \\ &= -(2nk+3)\frac{1}{3}\left\{2^n + (-1)^{n+1}\right\}, \\ \Theta_{\mathrm{C}_n}(a_g - a) &= -3\cdot 2^{n-1}\prod_{i=1}^{n-1}\left(-\frac{1}{2} - \zeta_n^{i(n-1)}\right) = -\left\{2^n + (-1)^{n+1}\right\}.\end{aligned}$$

Therefore, from Lemma 1, we obtain (5).

Lemma 6. *For any $k \in \mathbb{Z}$, $2^8(2k+1) \in S'(\mathrm{C}_4)$.*

Proof. Let $G = \mathrm{C}_4$ and let $a_0 = -k-2$, $a_1 = a_2 = -k-1$ and $a_3 = a = -k$. Then $\Theta'_G(a_g; a) = 2^8(2k+1)$.

Lemma 7. *For any $k \in \mathbb{Z}$, the following are elements of $S'(\mathrm{C}_2^2)$:*

(1) $8k+1$.

(2) $2^8(4k+1)$.

(3) $2^{12}(2k+1)$.

(4) $2^{14}k$.

Proof. Let $G = \mathrm{C}_2^2$ and let $b_0 := b_{(0,0)}$, $b_1 := b_{(1,0)}$, $b_2 := b_{(0,1)}$ and $b_3 := b_{(1,1)}$. If $b_0 = k+1$ and $b_1 = b_2 = b_3 = b = k$, then $\Theta'_G(b_g; b) = 8k+1$. If $b_0 = -k+1$, $b_1 = b_2 = b_3 = -k-1$ and $b = -k$, then $\Theta'_G(b_g; b) = 2^8(4k+1)$. If $b_0 = -k+1$, $b_1 = b_2 = -k-2$, $b_3 = -k-1$ and $b = -k$, then $\Theta'_G(b_g; b) = 2^{12}(2k+1)$. If $b_0 = -k-3$, $b_1 = -k-1$, $b_2 = b_3 = -k$ and $b = -k+1$, then $\Theta'_G(b_g; b) = 2^{14}k$.

Proof of Theorem 2 (1)–(5). We obtain (1) from $S'(\mathrm{C}_1) = \{x_0^2 - x^2 \mid x_0, x \in \mathbb{Z}\} = S(\mathrm{C}_2)$. We obtain (2) from Corollary 1 (2) and Lemma 5 (1)–(3). We obtain (3) from Corollary 1 (3) and Lemma 5 (1) and (3)–(5). We obtain (4) from Corollary 1 (4) and Lemmas 5 (1) and (3) and Lemma 6. We obtain (5) from Corollary 1 (5) and Lemma 7.

4. Proof of Theorem 2 (6)

To prove Theorem 2 (6), we use the following theorem, lemmas, and corollary.

Theorem 4 ([5, 8]). *Let p be a prime number. Then*

$$S(\mathrm{C}_p) = \mathbb{Z}_p^* \cup p^2\mathbb{Z}.$$

Here, $\mathbb{Z}_p^*$ denotes the set of all integers relatively prime to p.

Lemma 8. *We have*

$$\Theta_{\mathrm{C}_5}(x_g) \equiv \Theta_{\mathrm{C}_5}(x_g + 2x) \pmod{10}.$$

Proof. By direct computation, we see that there exists a polynomial $f(x_g, x)$ with integer coefficients such that

$$\Theta_{\mathrm{C}_5}(x_g) = \Theta_{\mathrm{C}_5}(x_g + 2x) + 10f(x_g, x).$$

This proves the lemma.

In Lemma 1, we set $G = \mathrm{C}_5$ and replace x_g with $x_g + x$. Then, together with Lemma 8, we obtain the following corollary.

Corollary 2. *The following identity holds:*

$$\Theta'_{\mathrm{C}_5}(x_g + x; x) = \Theta_{\mathrm{C}_5}(x_g)\,\Theta_{\mathrm{C}_5}(x_g + 2x) = \Theta_{\mathrm{C}_5}(x_g)\left(\Theta_{\mathrm{C}_5}(x_g) - 10f(x_g, x)\right).$$

Lemma 9. *For any $k, l \in \mathbb{Z}$, there exist $a_g, a \in \mathbb{Z}$ such that the following hold:*

(1) $\Theta_{\mathrm{C}_5}(a_g) = 5k + 1$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 1 + 10l$.

(2) $\Theta_{\mathrm{C}_5}(a_g) = 5k + 2$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 2 + 10l$.

(3) $\Theta_{\mathrm{C}_5}(a_g) = 5k + 3$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 3 + 10l$.

(4) $\Theta_{\mathrm{C}_5}(a_g) = 5k + 4$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 4 + 10l$.

(5) $\Theta_{\mathrm{C}_5}(a_g) = 5^2$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 5^2(2k + 1)$.

(6) $\Theta_{\mathrm{C}_5}(a_g) = 2 \cdot 5^2$ *and* $\Theta_{\mathrm{C}_5}(a_g + 2a) = 2 \cdot 5^2 k$.

Proof. If we set $a_0 = k + 1$, $a_1 = a_2 = a_3 = a_4 = k$, and $a = l$, then

$$\Theta_{\mathrm{C}_5}(a_g) = 5k + 1, \qquad \Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 1 + 10l.$$

If we set $a_0 = a_1 = k + 1$, $a_2 = a_3 = a_4 = k$, and $a = l$, then

$$\Theta_{\mathrm{C}_5}(a_g) = 5k + 2, \qquad \Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 2 + 10l.$$

If we set $a_0 = a_1 = a_2 = k + 1$, $a_3 = a_4 = k$, and $a = l$, then

$$\Theta_{\mathrm{C}_5}(a_g) = 5k + 3, \qquad \Theta_{\mathrm{C}_5}(a_g + 2a) = 5k + 3 + 10l.$$

If we set $a_0 = a_1 = a_2 = a_3 = k + 1$, $a_4 = k$, and $a = l$, then

$$\Theta_{C_5}(a_g) = 5k + 4, \qquad \Theta_{C_5}(a_g + 2a) = 5k + 4 + 10l.$$

If we set $a_0 = 2$, $a_1 = 0$, $a_2 = a_3 = a_4 = 1$, and $a = k$, then

$$\Theta_{C_5}(a_g) = 5^2, \qquad \Theta_{C_5}(a_g + 2a) = 5^2(2k + 1).$$

If we set $a_0 = 3$, $a_1 = 1$, $a_2 = a_3 = a_4 = 2$, and $a = k - 1$, then

$$\Theta_{C_5}(a_g) = 2 \cdot 5^2, \qquad \Theta_{C_5}(a_g + 2a) = 2 \cdot 5^2 k.$$

This completes the proof.

Theorem 2 (6) follows immediately from Theorem 4, Corollary 2, and Lemma 9.

We expect that the approach developed here can be applied to other explicit and structured specialization patterns of the group determinant, leading to further arithmetic information on $S(G)$.

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