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Lower estimates on the saturation order of approximation of twice continuously differentiable functions by piecewise constants on convex partitions

Розглядається задача про порядок наближення двічі неперервно диференційовних функцій багатьох змінних кусково-сталими сплайнами на опуклих розбиттях. Показано, що порядок насичення наближення кусково-сталими сплайнами в L_p нормі на опуклих розбиттях з N елементів становить $N^{-2/(d+1)}$, де d – число змінних.

Ключові слова: двічі неперервно диференційовні функції, кусково-сталі сплайни, порядок насичення, опуклі розбиття.

Рассматривается задача про порядок приближения дважды непрерывно дифференцируемых функций многих переменных кусочно-постоянными на выпуклых разбиениях. Показано, что порядок насыщения приближения кусочно-постоянными в L_p норме на выпуклых разбиениях с N элементами будет $N^{-2/(d+1)}$, где d – число переменных.

Ключевые слова: дважды непрерывно дифференцируемые функции, кусочно-постоянные сплайны, порядок насыщения, выпуклые разбиения.

We consider the problem of approximation order of twice continuously differentiable functions of many variables by piecewise constants. We show that the saturation order of piecewise constant approximation in L_p norm on convex partitions with N cells is $N^{-2/(d+1)}$, where d is the number of variables.

Key words: twice continuously differentiable functions, piecewise constant splines, saturation order, convex partitions.

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Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain. Let $L_p(\Omega)$, $1 \leq p \leq \infty$ be the space of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that $\|f\|_p$ is finite, where

$$\|f\|_p = \|f\|_{L_p(\Omega)} := \begin{cases} \left(\int_{\Omega} |f(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{esssup} \{ |f(\mathbf{x})| : \mathbf{x} \in \Omega \}, & p = \infty \end{cases}$$

For $1 \leq q \leq \infty$ and $r \in \mathbb{N}$, let $W_q^r(\Omega)$ be the standard Sobolev space of measurable functions $f : \Omega \rightarrow \mathbb{R}$ endowed with the standard norm

$$\|f\|_{W_q^r(\Omega)} := \sum_{\mathbf{k} \in \mathbb{Z}_+^d : |\mathbf{k}| \leq r} \|D^{\mathbf{k}} f\|_{L_q(\Omega)},$$

where $\mathbf{k} = (k_1, \dots, k_d)$ is the non-negative integer multi-index in $\mathbb{Z}_+^d$, $|\mathbf{k}| := k_1 + \dots + k_d$, and $D^{\mathbf{k}} = D_1^{k_1} \dots D_d^{k_d}$ is the partial differentiation operator, where $D_j^r = \frac{\partial^r}{\partial x_j^r}$ for $r \in \mathbb{Z}_+$.

A finite collection Δ of subdomains $\omega \subset \Omega$, referred to as *cells* in what follows, is called a *partition* of Ω provided that $\omega \cap \omega' = \emptyset$, for any $\omega, \omega' \in \Delta$, $\omega \neq \omega'$, and $\sum_{\omega \in \Delta} |\omega| = |\Omega|$, where $|\cdot|$ is the Lebesgue measure. We call a partition Δ *convex* if every cell $\omega \in \Delta$ is convex. For $N \in \mathbb{N}$, denote by $\mathfrak{D}_N$ the set of all convex partitions of Ω comprising at most N cells. For simplicity reason, we assume in this paper that Ω is a d -dimensional cube.

For a partition Δ of Ω , we denote by $\mathcal{S}_0(\Delta)$ the space of piecewise constant functions $s : \Omega \rightarrow \mathbb{R}$ that are constant on every cell $\omega \in \Delta$. For $1 \leq p \leq \infty$, we define the error of the best L_p -approximation of a function $f \in L_p(\Omega)$ by piecewise constant functions on convex partitions from $\mathfrak{D}_N$:

$$E_N(f)_p := \inf_{\Delta \in \mathfrak{D}_N} \inf_{s \in \mathcal{S}_0(\Delta)} \|f - s\|_{L_p(\Omega)}.$$

It was established by Birman and Solomyak (see Theorems 3.1 and 3.2 in [1]), that for every $f \in W_q^1(\Omega)$,

$$E_N(f)_p = \mathcal{O}(N^{-1/d}) \quad \text{as } N \rightarrow \infty, \quad (1)$$

as soon as $\frac{1}{d} + \frac{1}{p} - \frac{1}{q} > 0$. The construction of a suitable sequence of partitions $\{\Delta_N\}_{N=1}^\infty$ that achieves this order is based on adaptively refined dyadic subdivisions of the cube Ω . Since $W_q^2(\Omega)$ is imbedded into $W_{q'}^1(\Omega)$ as soon as parameter $1 \leq q' \leq \infty$ satisfies inequality $\frac{1}{d} + \frac{1}{q'} - \frac{1}{q} > 0$, it follows from (1) that, for a function $f \in W_q^2(\Omega)$,

$$E_N(f)_p = \mathcal{O}(N^{-1/d}) \quad \text{as } N \rightarrow \infty, \quad (2)$$

providing that $\frac{2}{d} + \frac{1}{p} - \frac{1}{q} > 0$. No improvement of the order $\mathcal{O}(N^{-1/d})$ can be achieved for a smooth non-constant function on “isotropic” partitions (see [2]).

It was shown in [2, 3] that on a wider set of all convex partitions $\mathfrak{D}_N$ essentially better order of approximation is possible. More precisely, for every $1 \leq p \leq \infty$ and $q = p$, and every function $f \in W_q^2(\Omega)$,

$$E_N(f)_p = \mathcal{O}(N^{-2/(d+1)}) \quad \text{as } N \rightarrow \infty,$$

which almost doubles the approximation order in comparison to $N^{-1/d}$ when d is large enough. This improvement in order is achieved on “anisotropic” sequence of partitions $\{\Delta_N^*\}_{N=1}^\infty$ that is constructed in the following way:

1. First, the cube Ω is subdivided into m^d equal cubes ω_i , $i = 1, \dots, m^d$, of side length $\frac{1}{m}$, where $m = \lceil N^{1/(d+1)} \rceil$;
2. Next, each of ω_i 's is split into m slices ω_{ij} , $j = 1, \dots, m$, by equidistant hyperplanes orthogonal to the average gradient $\frac{1}{|\omega_i|} \int_{\omega_i} \nabla f(\mathbf{x}) d\mathbf{x}$ on ω_i , where $\nabla f(\mathbf{x})$ is the gradient of function f at the point $\mathbf{x}$.

Moreover, Davydov [3] has shown that $N^{-2/(d+1)}$ is the saturation order of piecewise constant approximation on convex partitions in the following sense: *if for a twice continuously differentiable function $f : \Omega \rightarrow \mathbb{R}$, we have that $E_N(f)_\infty = o(N^{-2/(d+1)})$ as $N \rightarrow \infty$, then the Hesse matrix of function f is neither positive definite nor negative definite on Ω .* We remark that similar results in the case $d = 2$ were obtained by a different method in [4].

Recently in [5] it was established that for $1 \leq p, q \leq \infty$ which satisfy inequality $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geq 0$ and for all functions $f \in W_q^2(\Omega)$,

$$E_N(f)_p = \mathcal{O}(N^{-2/(d+1)}) \quad \text{as } N \rightarrow \infty.$$

In this paper we will show that $N^{-2/(d+1)}$ is in fact a saturation order in this case. Let us introduce some additional notation.

Let $C^2(\Omega)$ be the space of twice continuously differentiable functions $f : \Omega \rightarrow \mathbb{R}$.

Denote *Euclidean distance* as $\|\cdot\|_2$.

The *Hesse matrix* of $f \in C^2(\Omega)$ is defined as follows

$$\nabla^2 f(\mathbf{x}) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_d \partial x_d}(\mathbf{x}) \end{pmatrix},$$

where $\mathbf{x} \in \Omega$. It's well-known (see [6]), that the Hesse matrix of a convex function is positive definite, and is negative definite for a concave function. For simplicity we will consider only case when the Hesse matrix of function is positive definite. However the presented arguments could be easily adjusted for negative definite matrices.

The main result of this paper is the following

Theorem 1. *Let $d \in \mathbb{N}$, $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain, $f \in C^2(\Omega)$ be such that there exists a point $\tilde{\mathbf{x}} \in \Omega$ where $\nabla^2 f(\tilde{\mathbf{x}})$ is positive definite. Then there exists a constant $C = C(d, f, p, \Omega)$ such that*

$$E_N(f)_p \geq CN^{-\frac{2}{d+1}}.$$

This result shows that $N^{-2/(d+1)}$ is the saturation order of piecewise constant approximation on convex partitions in the following sense: *if for a twice continuously differentiable function $f : \Omega \rightarrow \mathbb{R}$, we have that $E_N(f)_p = o(N^{-2/(d+1)})$ as $N \rightarrow \infty$, then the Hesse matrix of function f is neither positive, nor negative definite on Ω .*

To prove Theorem 1 we need several auxiliary results.

The assertions of geometrical Lemmas 1 and 2 below are intuitively clear, and we give their proofs for the completeness of the exposition.

Lemma 1. *Let T be a simplex in $\mathbb{R}^d$, V is a volume of this simplex, a is the length of one of the side of the simplex, and S is a volume of projection of the simplex T on the hyperplane orthogonal to the side a . Then*

$$V = \frac{1}{d} \cdot a \cdot S.$$

Proof. Let us show how this theorem will look for triangle. And then will prove it in general case. For $\mathbb{R}^2$, we need to prove that $S_{\Delta ABC} = \frac{1}{2} \cdot a \cdot S$, where a is length of the side AB , and S is a length of projection of triangle ABC on the line orthogonal to AB . We can use side AB as Oy axis, and orthogonal line passing through the point A as Ox axis of coordinates. Then we have:

$$S_{\Delta ABC} = \int_S \int_{AC(x)}^{BC(x)} 1 dy dx = \int_S \int_0^{BC(x)-AC(x)} 1 dy dx,$$

where $AC(x)$ and $BC(x)$ are equations of lines AC and BC correspondingly. In geometrical sense this means that the area of the triangle doesn't change if we move vertex C along a line parallel to AB . So we can move C , untill side CA coincides with the projection of triangle ABC on Ox axis. And the area of this triangle could be counted by formula $S_{\Delta ABC} = \frac{1}{2} \cdot a \cdot S$.

Let $A_1 A_2 \dots A_{d+1}$ be a simplex in $\mathbb{R}^d$. Without lose of generality we can choose side $A_d A_{d+1}$ and hyperplane which is orthogonal to this side and pass through vertex A_d . Let $B_1 B_2 \dots B_{d-1}$ be the projection of vertices $A_1, A_2, \dots, A_{d-1}$ correspondingly. We can use side $A_d A_{d+1}$ as Ox_1 axis. And other axes will be in orthogonal hyperplane. Then we have

$$\begin{aligned} V_{A_1 A_2 \dots A_{d+1}} &= \int_{B_1 B_2 \dots B_{d-1} A_d} \int_{A_1 A_2 \dots A_{d-1} A_d(x_2, \dots, x_d)}^{A_1 A_2 \dots A_{d-1} A_{d+1}(x_2, \dots, x_d)} 1 dx_1 dx_2 \dots dx_d \\ &= \int_{B_1 B_2 \dots B_{d-1} A_d} \int_0^{A_1 A_2 \dots A_{d-1} A_{d+1}(x_2, \dots, x_d) - A_1 A_2 \dots A_{d-1} A_d(x_2, \dots, x_d)} 1 dx_1 dx_2 \dots dx_d. \end{aligned}$$

In geometrical sense this means that the volume of the simplex doesn't change if we move vertices $A_1, A_2, \dots, A_{d-1}$ along a line parallel to $A_d A_{d+1}$. So we can move them, they coincides with the projection of simplex $A_1 A_2 \dots A_{d+1}$ on hyperplane. And the volume of this simplex could be counted by formula $V = \frac{1}{d} \cdot |A_d A_{d+1}| \cdot S$.

The following theorem is some kind of generalization of Lemma 1, and it shows how connected volume of arbitrary convex domain Ω with volume of orthogonal projection Ω' . Let $\bar{v}$ is arbitrary direction vector in $\mathbb{R}^d$. Denote $h(\Omega, \bar{v})$ the length of the longest segment inside domain Ω , which is parallel to vector $\bar{v}$.

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain. Let $h(\Omega, \bar{v})$ be the length of the longest segment in Ω in given direction $\bar{v}$, and $|\Omega'|$ be the volume of projection of the domain Ω on $(d-1)$ -dimensional hyperplane orthogonal to $\bar{v}$. Then*

$$\frac{1}{d!} \cdot h(\Omega, \bar{v}) \cdot |\Omega'| \leq |\Omega| \leq h(\Omega, \bar{v}) \cdot |\Omega'|. \quad (3)$$

Proof. Let us find the longest segment in domain $\Omega \subset \mathbb{R}^2$ parallel to vector $\bar{v}$, and call it A_1A_2 . Project domain Ω on line orthogonal to A_1A_2 . Let B_1 and B_2 be the ends of the segment of projection. And B – projection of the A_1A_2 . Denote on Ω the points C_1 and C_2 that correspond to the points B_1 and B_2 on the projection. If there are several point which correspond to B_1 and B_2 on the projection, then we can choose anyone. As Ω is convex, that means that polygon $A_1C_1A_2C_2$ is completely inside Ω . Therefore

$$|\Omega| \geq S_{A_1C_1A_2C_2}.$$

According to Lemma 1 we have

$$S_{A_1C_1A_2C_2} = S_{A_1C_1A_2} + S_{A_1A_2C_2} = \frac{1}{2}|A_1A_2| \cdot |B_1B| + \frac{1}{2}|A_1A_2| \cdot |B_2B| = \frac{1}{2}|A_1A_2| \cdot |B_1B_2|.$$

From the other side length of every segment of Ω parallel to $\bar{v}$ is less then $|A_1A_2|$. Then $|\Omega|$ is less then volume of rectangle with sides A_1A_2 and B_1B_2 . So we got following inequalities

$$\frac{1}{2!} \cdot h(\Omega, \bar{v}) \cdot |\Omega'| \leq |\Omega| \leq h(\Omega, \bar{v}) \cdot |\Omega'|,$$

for $\Omega \subset \mathbb{R}^2$. Let the described inequalities holds for $\Omega \subset \mathbb{R}^{d-1}$.

For $\Omega \subset \mathbb{R}^d$ let us find the longest segment parallel to vector $\bar{v}$, and call it A_1A_2 . Project domain Ω on hyperplane α_1 , which orthogonal to A_1A_2 and pass through A_1 . We get $\Omega' \subset \mathbb{R}^{d-1}$ – projection of Ω . In hyperplane α_1 we can choose any direction $\bar{v}_1$. Let $A_{11}A_{12}$ be the longest segment in Ω' parallel to vector $\bar{v}_1$. Then project domain Ω' on hyperplane $\alpha_2 \subset \mathbb{R}^{d-2}$, where α_2 is orthogonal to $\bar{v}_1$, and $\alpha_2 \subset \alpha_1$. We get domain $\Omega'' \subset \mathbb{R}^{d-2}$ – projection of the Ω' on hyperplane α_2 . Inside hyperplane α_2 we choose any direction vector $\bar{v}_2$. Denote the longest segment of Ω'' parallel to vector $\bar{v}_2$ as $A_{21}A_{22}$. And so on. When we will get the projection $\Omega^{(d-1)}$ this will be just a segment $A_{d-11}A_{d-12} \subset \mathbb{R}$. Then we denote points on the projections of higher dimensionality points which corresponds to points A_{k1}, A_{k2} for $k = 1 \dots d-1$. On domain Ω we denote points B_{k1}, B_{k2} which corresponds to points A_{k1}, A_{k2} for $k = 1 \dots d-1$. We will get convex polyhedron with vertices A_1, A_2 and B_{k1}, B_{k2} for $k = 1 \dots d-1$, which is inscribed into Ω . We separate this polyhedron on d -dimension simplexes of the form $A_1A_2A_{1i_1}A_{2i_2} \dots A_{d-1i_{d-1}}$, where i_k is 1 or 2 for $k = 1, \dots, d-1$. The total number of such simplexes will be 2^{d-1} . Projection of this polyhedron on Ω' will be polyhedron inscribed into Ω' with vertices A_{11}, A_{12} and points which corresponds to points A_{k1}, A_{k2} from $\Omega^{(k)}$ on Ω' . Let V_1 be the volume of d -dimension polyhedron. According to Lemma

1 we have:

$$\begin{aligned}
 V_1 &= \sum_{(i_1, \dots, i_{d-1}) \in \{1, 2\}^{d-1}} V_{A_1 A_2 B_{1i_1} \dots B_{d-1i_{d-1}}} \\
 &= \sum_{(i_1, \dots, i_{d-1}) \in \{1, 2\}^{d-1}} \frac{1}{d} \cdot |A_1 A_2| \cdot V'_{A_1 A_2 B_{1i_1} \dots B_{d-1i_{d-1}}} \\
 &= \frac{1}{d} \cdot |A_1 A_2| \cdot \sum_{(i_1, \dots, i_{d-1}) \in \{1, 2\}^{d-1}} V'_{A_1 A_2 B_{1i_1} \dots B_{d-1i_{d-1}}},
 \end{aligned}$$

where $V'_{A_1 A_2 B_{1i_1} \dots B_{d-1i_{d-1}}}$ is volume of projection of simplex with vertices $A_1 A_2 B_{1i_1} \dots B_{d-1i_{d-1}}$. This sum is a volume of polyhedron inscribed in Ω' with vertices A_{11}, A_{12} and points which corresponds to points A_{k1}, A_{k2} from $\Omega^{(k)}$ on Ω' . According to the inductive hypothesis, we obtain

$$V_1 = \frac{1}{d!} \cdot |A_1 A_2| \cdot |A_{11} A_{12}| \cdot \dots \cdot |A_{d-11} A_{d-12}|,$$

where each segment $A_{k1} A_{k2}$ is the longest segment in $\Omega^{(k)}$ in direction of vector $\overline{v_k}$. According to the inductive hypothesis, we have

$$\begin{aligned}
 |A_{d-11} A_{d-12}| &= |\Omega^{(d-1)}|, \\
 |A_{d-11} A_{d-12}| |\Omega^{(d-1)}| &\geq |\Omega^{(d-2)}|, \\
 &\dots \\
 |A_{11} A_{12}| |\Omega''| &\geq |\Omega'|.
 \end{aligned}$$

Therefore

$$|\Omega| \geq V_1 \geq \frac{1}{d!} \cdot |A_1 A_2| \cdot |\Omega'|.$$

From the other side we have

$$|\Omega| = \int_{\Omega'} AB(x_1, x_2, \dots, x_{d-1}) dx_1 dx_2 \dots dx_{d-1} \leq \int_{\Omega'} |A_1 A_2| dx_1 dx_2 \dots dx_{d-1} = |A_1 A_2| |\Omega'|,$$

here $AB(x_1, x_2, \dots, x_{d-1})$ is length of the segment, which is the result of intersection of line which pass through point $(x_1, x_2, \dots, x_{d-1})$ in projection Ω' with domain Ω .

Lemma 3. *Let $f \in C^2((a; b))$ and $\gamma, \delta > 0$ are such that $|f'(x)| \geq \gamma$ and $|f''(x)| \geq \delta$, for every $x \in (a; b)$. Then following inequalities hold*

$$E_1(f)_1 \geq \frac{(b-a)^2}{4} \cdot \gamma, \tag{4}$$

$$E_1(f)_1 \geq \frac{(b-a)^3}{16} \cdot \delta. \tag{5}$$

Proof. Without loss of generality let f be increasing and convex function. The constant of best approximation in L_1 on interval $(a; b)$ is $f\left(\frac{a+b}{2}\right)$ (see [7]). Therefore

$$\begin{aligned} E_1(f)_1 &= \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right) \right| dx = \int_{\frac{a+b}{2}}^b \left| f(t) - f\left(t - \frac{b-a}{2}\right) \right| dt \\ &= \int_{\frac{a+b}{2}}^b \int_{t-\frac{b-a}{2}}^t |f'(u)| du dt \geq \int_{\frac{a+b}{2}}^b \int_{t-\frac{b-a}{2}}^t \gamma du dt = \gamma \left(\frac{b-a}{2}\right)^2. \end{aligned}$$

Also for $f'(u)$ we have the following inequality

$$f'(u) = f'\left(t - \frac{b-a}{2}\right) + \int_{t-\frac{b-a}{2}}^u f''(\tau) d\tau \geq \int_{t-\frac{b-a}{2}}^u \delta d\tau = \delta \cdot \left(u - \left(t - \frac{b-a}{2}\right)\right).$$

Therefore

$$\begin{aligned} E_1(f)_1 &= \int_{\frac{a+b}{2}}^b \int_{t-\frac{b-a}{2}}^t |f'(u)| du dt \geq \delta \cdot \int_{\frac{a+b}{2}}^b \int_{t-\frac{b-a}{2}}^t \left(u - \left(t - \frac{b-a}{2}\right)\right) du dt \\ &= \delta \cdot \int_{\frac{a+b}{2}}^b \frac{\left(\frac{b-a}{2}\right)^2}{2} dt = \delta \cdot \frac{(b-a)^3}{16}. \end{aligned}$$

The following theorem is generalization of Lemma 3 for functions defined on $\mathbb{R}^d$.

Lemma 4. Let $f \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^d$. Let exists direction vector σ , that $D_\sigma f(\mathbf{x}) \geq \gamma > 0$, for every $\mathbf{x} \in \Omega$. The following inequality holds

$$E_1(f)_1 \geq \frac{1}{4} \cdot \frac{1}{d!} \cdot \gamma \cdot |\Omega| \cdot h(\Omega, \sigma), \quad (6)$$

where $h(\Omega, \sigma)$ was defined before Lemma 2.

Moreover assume that the Hesse matrix of f is positive definite on Ω , and $\delta > 0$ is such that smallest eigenvalue of $\nabla^2 f$ is at least δ everywhere on Ω . Then

$$E_1(f)_1 \geq \frac{1}{16} \cdot \left(\frac{1}{d!}\right)^2 \cdot \delta \cdot |\Omega| \cdot \text{diam}^2(\Omega), \quad (7)$$

where $\text{diam}(\Omega)$ is the longest segment inside Ω .

Proof. We can change orthonormal basis in $\mathbb{R}^d$ in such way, that direction of σ and axis Ox_1 coincide. Then according to (4) from Lemma 3 we have

$$\begin{aligned} E_1(f)_1 &= \inf_{c \in \mathbb{R}} \int_{\Omega} |f(\mathbf{x}) - c| d\mathbf{x} = \inf_{c \in \mathbb{R}} \int_{\Omega'} dx_2 \dots dx_d \int_{a(x_2, \dots, x_d)}^{b(x_2, \dots, x_d)} |f(x_1, \dots, x_d) - c| dx_1 \\ &\geq \int_{\Omega'} \frac{(b(x_2, \dots, x_d) - a(x_2, \dots, x_d))^2}{4} \cdot \gamma dx_2 \dots dx_d, \end{aligned}$$

where Ω' is projection of Ω on the coordinate hyperplane orthogonal to σ , $a(x_2, \dots, x_d)$ and $b(x_2, \dots, x_d)$ are the ends of the segment, which is the result of intersection of the line, which pass through point $(x_2, \dots, x_d) \in \Omega'$ and parallel to σ , with domain Ω .

According to Hölder's inequality we obtain

$$\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})| d\mathbf{t} \leq \left(\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})|^2 d\mathbf{t} \right)^{\frac{1}{2}} \left(\int_{\Omega'} 1 d\mathbf{t} \right)^{\frac{1}{2}}.$$

According to lower estimate (3) in Lemma 2 we have

$$\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})|^2 d\mathbf{t} \geq \frac{\left(\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})| d\mathbf{t} \right)^2}{\int_{\Omega'} 1 d\mathbf{t}} = \frac{|\Omega|^2}{|\Omega'|} \geq \frac{1}{d!} \cdot |\Omega| \cdot h(\Omega, \sigma).$$

Therefore

$$E_1(f)_1 \geq \frac{1}{4} \cdot \frac{1}{d!} \cdot \gamma \cdot |\Omega| \cdot h(\Omega, \sigma).$$

The first part of theorem is proved. For the second part we can change orthonormal basis in $\mathbb{R}^d$ in such way, that direction of $diam$ and axis Ox_1 coincide. Then according to (5) from Lemma 3 we have

$$\begin{aligned} E_1(f)_1 &= \inf_{c \in \mathbb{R}} \int_{\Omega} |f(\mathbf{x}) - c| d\mathbf{x} = \inf_{c \in \mathbb{R}} \int_{\Omega'} dx_2 \dots dx_d \int_{a(x_2, \dots, x_d)}^{b(x_2, \dots, x_d)} |f(x_1, \dots, x_d) - c| dx_1 \\ &\geq \int_{\Omega'} \frac{(b(x_2, \dots, x_d) - a(x_2, \dots, x_d))^3}{16} \cdot \delta dx_2 \dots dx_d, \end{aligned}$$

where Ω' is projection of Ω on the coordinate hyperplane orthogonal to $diam(\Omega)$, $a(x_2, \dots, x_d)$ and $b(x_2, \dots, x_d)$ are the ends of the segment, which is the result of intersection of the line, which pass through point $(x_2, \dots, x_d) \in \Omega'$ and parallel to $diam(\Omega)$, with domain Ω .

According to Hölder's inequality we obtain

$$\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})| d\mathbf{t} \leq \left(\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})|^3 d\mathbf{t} \right)^{\frac{1}{3}} \left(\int_{\Omega'} 1 d\mathbf{t} \right)^{\frac{2}{3}}.$$

According to lower estimate (3) in Lemma 2 we have

$$\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})|^3 d\mathbf{t} \geq \frac{\left(\int_{\Omega'} |b(\mathbf{t}) - a(\mathbf{t})| d\mathbf{t} \right)^3}{\left(\int_{\Omega'} 1 d\mathbf{t} \right)^2} = \frac{|\Omega|^3}{|\Omega'|^2} \geq \left(\frac{1}{d!} \right)^2 \cdot |\Omega| \cdot \text{diam}^2(\Omega).$$

Therefore

$$E_1(f)_1 \geq \frac{1}{16} \cdot \left(\frac{1}{d!} \right)^2 \cdot \delta \cdot |\Omega| \cdot \text{diam}^2(\Omega).$$

Using the results of Lemma 2 and Lemma 4 we can prove Theorem 1.

Proof of Theorem 1. Let us prove the theorem for case $p = 1$. First choose $\delta > 0$ and convex domain $Q \subset \Omega$ such that the smallest eigenvalue of the Hessian of f is at least δ everywhere in Q . Clearly, $\nabla f(\tilde{\mathbf{x}}) \neq 0$, and $\tilde{\mathbf{x}} \in Q$. Since the gradient of f is continuous, there is a constant γ and convex domain $\tilde{Q} \subset Q$, $\tilde{Q} \ni \tilde{\mathbf{x}}$, that $D_\sigma f(\mathbf{x}) \geq \gamma$, for every $\mathbf{x} \in \tilde{Q}$, where $\sigma = \frac{\nabla f(\tilde{\mathbf{x}})}{\|\nabla f(\tilde{\mathbf{x}})\|_2}$. We assume without loss of generality that $\tilde{Q} = Q$.

According to (7) in Lemma 4

$$E_1(f)_{L_1(\omega)} \geq E_1(f)_{L_1(\omega \cap Q)} \geq \frac{1}{16} \cdot \left(\frac{1}{d!} \right)^2 \cdot \delta \cdot |\omega \cap Q| \cdot \text{diam}^2(\omega \cap Q),$$

for every $\omega \in \Delta$ and $\omega \cap Q \neq \emptyset$. Therefore

$$\text{diam}(\omega \cap Q) \leq \left(\frac{16 \cdot (d!)^2}{\delta \cdot |\omega \cap Q|} \cdot E_1(f)_{L_1(\omega)} \right)^{\frac{1}{2}}.$$

Also, according to (6) in Lemma 4

$$E_1(f)_{L_1(\omega)} \geq E_1(f)_{L_1(\omega \cap Q)} \geq \frac{1}{4} \cdot \frac{1}{d!} \cdot \gamma \cdot |\omega \cap Q| \cdot h(\omega \cap Q, \sigma),$$

for every $\omega \in \Delta$ and $\omega \cap Q \neq \emptyset$. Therefore

$$h(\omega \cap Q, \sigma) \leq \frac{4 \cdot d!}{\gamma \cdot |\omega \cap Q|} \cdot E_1(f)_{L_1(\omega)}.$$

If we project $\omega \cap Q$ on hyperplane orthogonal to σ , the projection will be contained inside $(d-1)$ -dimensional ball of radius $\left(\frac{16 \cdot (d!)^2}{\delta \cdot |\omega \cap Q|} \cdot E_1(f)_{L_1(\omega)}\right)^{\frac{1}{2}}$. According to upper estimate in (3) in Lemma 2 we have

$$|\omega \cap Q| \leq \frac{4 \cdot d!}{\gamma \cdot |\omega \cap Q|} \cdot E_1(f)_{L_1(\omega)} \cdot \mu_{d-1} \cdot \left(\frac{16 \cdot (d!)^2}{\delta \cdot |\omega \cap Q|} \cdot E_1(f)_{L_1(\omega)}\right)^{\frac{d-1}{2}},$$

where μ_{d-1} denotes volume of $(d-1)$ -dimensional ball of radius 1. Therefore

$$\begin{aligned} |\omega \cap Q|^{\frac{d+3}{2}} &\leq \frac{4^d \cdot (d!)^d \cdot \mu_{d-1}}{\gamma \cdot \delta^{\frac{d-1}{2}}} \cdot E_1(f)_{L_1(\omega)}^{\frac{d+1}{2}} = C_1 \cdot E_1(f)_{L_1(\omega)}^{\frac{d+1}{2}}, \\ |\omega \cap Q|^{\frac{d+3}{d+1}} &\leq C_2 \cdot E_1(f)_{L_1(\omega)}, \end{aligned}$$

where constants $C_1 = C_1(f, d)$ and $C_2 = C_2(f, d)$.

Summing over all $\omega \in \Delta$ we obtain

$$\sum_{\omega \in \Delta} |\omega \cap Q|^{\frac{d+3}{d+1}} \leq C_2 \cdot E_N(f)_1.$$

According to Hölder's inequality we have

$$|Q| = \sum_{\omega \in \Delta} |\omega \cap Q| \leq \left(\sum_{\omega \in \Delta} |\omega \cap Q|^{\frac{d+3}{d+1}} \right)^{\frac{d+1}{d+3}} \cdot N^{\frac{2}{d+3}} \leq C_3 \cdot E_N(f)_1^{\frac{d+1}{d+3}} \cdot N^{\frac{2}{d+3}},$$

where constant $C_3 = C_3(f, d)$. And therefore

$$E_N(f)_1 \geq C_4 \cdot N^{-\frac{2}{d+1}},$$

where constant $C_4 = C_4(f, d, \Omega)$.

According to Hölder's inequality, for any arbitrary $1 < p < \infty$ we have

$$E_N(f)_1 \leq E_N(f)_p \cdot \left(\int_{\Omega} 1 d\mathbf{x} \right)^{1-\frac{1}{p}} = E_N(f)_p \cdot |\Omega|^{1-\frac{1}{p}}.$$

Therefore

$$E_N(f)_p \geq E_N(f)_1 \cdot |\Omega|^{-(1-\frac{1}{p})} \geq C_5 \cdot N^{-\frac{2}{d+1}},$$

where constant $C_5 = C_5(f, d, p, \Omega)$.

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References

1. *Birman M. S.* Piecewise polynomial approximation of functions of the classes W_p^α [Text] /M. S. Birman, M. Z. Solomyak // Mat. Sb. – 1967. Vol. 73 (115), no. 3. – pp. 331–355 (in Russian). English translation in Math. USSR-Sb. – 1967. – Vol. 2, no. 3. – pp. 295–317.

2. *Davydov O.* Algorithms and error bounds for multivariate piecewise constant approximation [Text]/O. Davydov // in Approximation Algorithms for Complex Systems, (E. H. Georgoulis, A. Iske and J. Levesley, Eds.), Springer Proceedings in Mathematics. – 2011. – Vol. 3. – pp. 27–45.
3. *Davydov O.* Approximation by piecewise constants on convex partitions [Text] /O. Davydov // J. Approx. Theory. – 2012. – Vol. 164. – pp. 346–352.
4. *Kochurov A. S.* Approximation by piecewise constant functions on the square [Text] /A. S. Kochurov // East J. Approx. – 1995. – Vol. 1. – pp. 463–478.
5. *Kozynenko O.* Adaptive anisotropic approximation of multivariate functions by piecewise constants [Text]/O. Kozynenko, O. Davydov, D. Skorokhodov// International Conference of Young Mathematicians dedicated to the 100th Anniversary of Academician of National Academy of Sciences of Ukraine, Professor Yu. O. Mitropolskiy (1917–2008): int. conf. (Kyiv, Ukraine, June 7-10, 2017), Abstracts. – Kyiv:Institute of Mathematics of NAS of Ukraine. – 2017. – p. 34.
6. *Kudryavcev L. D.* A brief course of mathematical analysis [Text]/ L. D. Kudryavcev. – M.: Fizmatlit, 2005. – 424 p.
7. *Korneichuk N. P.* Exact constants in approximation theory [Text]/ N. P. Korneichuk. – M.: Nauka, 1987. – 424 p.

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