The uniqueness of the best non-symmetric $L_1$-approximant with a weight by $A_{\alpha,\beta}$-subspace

The questions of the uniqueness of the best non-symmetric $L_1$-approximant with weight in the finite dimensional subspace and the connection of such tasks with $A_{\alpha,\beta}$-subspaces were considered in this article. This result generalizes the known result of Kroo on the case of non-symmetric approximation.
Let $X$ be a partially ordered set and its order is consistent with algebraic operations. The following definitions are given in [5].

Let $E \subset X$ be a non-empty set. The element $y \in X$ is called supremum (infimum) of the set $E$ and is denoted by $\sup E$ ($\inf E$) if the following conditions hold:

1) $x \leq y$ ($x \geq y$) $\forall x \in E$;

2) for any element $z \in X$ such that $x \leq z$ ($x \geq z$), it follows that $y \leq z$ ($y \geq z$).

The supremum of the set $E$ is denoted by $x_1 \vee x_2 \vee ... \vee x_n$ and the infimum of the set $E$ is denoted by $x_1 \wedge x_2 \wedge ... \wedge x_n$ if the set $E$ consists of elements $x_1, x_2, ..., x_n$.

Suppose in the space $X$ for any two elements $x, y \in X$ there exists its supremum $x \vee y$; then the element $x_+ = x \vee 0$ is called the positive part of the element $x \in X$, the element $x_- = (-x) \vee 0$ is its negative part, and the element $|x| = x_+ + x_-$ is the module of the element $x$. Two elements $x, y \in X$ are called disjunctive and are denoted by $x \Delta y$ if $|x| \cap |y| = 0$.

Let an order of a partially ordered vector space $X$ is consistent with algebraic operations and for any two elements $x, y \in X$ there exists its supremum $x \vee y$. Then a space $X$ is called a KN-lineal if in $X$ the monotone norm is defined, i. e., $|x| \leq |y| \Rightarrow ||x||x \leq ||y||x$.

A KN-lineal is called a KN-space (or $K_{\sigma}N$-space) if for any (or any numbered) non-empty set bounded above or below there exists the its upper or lower bound respectively.

Let $X$ is a KN-linear and $X^*$ is a space of linear continuous in the usual sense functionals on $X$, then $X^*$ is a complete KN-space in the usual sense.

A $K_{\sigma}N$-space is called a KB-space if its norm satisfies two conditions:

1) $||x_n||_X \to 0$ if $x_n \downarrow 0$;

2) $||x_n||_X \to +\infty$ if $x_n \uparrow +\infty$ ($x_n \geq 0$).

Let $K$ be a compact subset of $\mathbb{R}^m$ such that $K = \overline{intK}$, $\mu$ be the Lebesgue measure in $\mathbb{R}^m$ and $\mu(intK) > 0$.

Let $X$ be a KB-space with the norm $|| \cdot ||_X$.

By $C(K, X)$ denote the space of continuous functions $f : K \to X$ and by $\Theta$ denote the set of all $\mu$-measurable real functions $\theta$ on $K$ such that $0 < \inf \{\theta(x) : x \in K\} \leq \sup \{\theta(x) : x \in K\} < \infty$.

For any $x \in K$ and positive numbers $\alpha, \beta$ put

$|f(x)|_{\alpha, \beta} = \alpha \cdot f_+(x) + \beta \cdot f_-(x),$

$||f(x)||_{X, \alpha, \beta} = ||\alpha \cdot f_+(x) + \beta \cdot f_-(x)||_X,$

where $f_\pm(x) = (\pm f(x)) \vee 0$.

Suppose the space $C(K, X)$ is supplied with the non-symmetric $L_1$-norm with a weight $\theta \in \Theta$:

$||f||_{1; \theta, \alpha, \beta} = \int_K \theta(x)||f(x)||_{X, \alpha, \beta}d\mu(x),$
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and denote by $C_\theta(K, X)$ the space $C(K, X)$ endowed with the above norm.

For $f \in C_\theta(K, X)$, $H \subset C_\theta(K, X)$ and $\theta \in \Theta$ the quantity

$$E(f, H)_{1; \theta; \alpha, \beta} = \inf_{g \in H} \|f - g\|_{1; \theta; \alpha, \beta}$$

(1)

is called the best $(\alpha, \beta)$-approximation of a function $f$ by a set $H$ in the metric $L_1$ with a weight $\theta$. The function $g^* \in H$ is the best $(\alpha, \beta)$-approximant of a function $f$ by elements of a set $H$ in the metric $L_1$ with a weight $\theta$ if $g^*$ realizes the greatest lower bound in the equality (1). By $Z_f$ denote the set of zeros for a function $f$, and $N_f = K \setminus Z_f$.

For $f, g \in C_\theta(K, X)$ put

$$\tau_{\pm}^{(\alpha, \beta)}(f, g)_{1, \theta} = \lim_{t \to \pm 0} \frac{\|f + tg\|_{1; \theta; \alpha, \beta} - \|f\|_{1; \theta; \alpha, \beta}}{t}$$

and for $x \in K$ put

$$\tau_{\pm}^{(\alpha, \beta)}(f(x), g(x))_X = \lim_{t \to \pm 0} \frac{||(f + tg)(x)||_{X; \alpha, \beta} - ||f(x)||_{X; \alpha, \beta}}{t}.$$ 

For $\alpha = \beta = 1$ such functionals were considered in [2] and [1].

A normalized space $X$ is called strictly convex if for any $x, y \in X$ such that $||x + y|| = ||x|| + ||y||$, it follows that there is $\lambda \in \mathbb{R}$ such that $y = \lambda \cdot x$.

A KB-space $X$ is the space with a strictly monotone norm if $|x| < |y| \Rightarrow ||x||_X < ||y||_X$.

Let $H$ be a subspace of the space $C(K, X)$. We set

$$H' = \{h \in C(K, X) : \exists g_h \in H \forall x \in K \ h(x) = \pm g_h(x)\}.$$ 

Such classes were considered in [1]. Originally such sets were introduced by Hans Strauss [3] for $X = \mathbb{R}$, $K = [a, b]$.

The following theorem (see [4]) is needed for the sequel.

**Theorem 1.** Let $X$ be a KB-space, $H$ be a subspace of space $C_\theta(K, X)$, $\theta \in \Theta$. Then in order that each function $f \in C_\theta(K, X)$ has at most one the best $(\alpha, \beta)$-approximant with a weight $\theta \in \Theta$ by elements from $H$ it is necessary that for any function $h \in H \setminus \{0\}$ there exists a function $g_0 \in H$ such that

$$\int_{N_h} \theta(x) \tau_{\pm}^{(\alpha, \beta)}(h(x), g_0(x))_X d\mu(x) > \int_{Z_h} \theta(x) ||g_0(x)||_{X; \alpha, \beta} d\mu(x).$$

If the KB-space $X$ is strictly convex with a strictly monotone norm then this condition is sufficient.

The finite dimensional subspace $H \subset C(K, X)$ is called an $A_{\alpha, \beta}$-subspace (or is said to satisfy the $A_{\alpha, \beta}$-property) if for any $h \in H' \setminus \{0\}$ there exists a $g \in H$ such that

(i) $g(x) = 0$ a.e. on $Z_h$;

(ii) $\tau_{-}^{(\alpha, \beta)}(h(x), g(x))_X \geq 0$ a.e. on $N_h$ and this inequality is strict on a subset of $N_h$ of positive measure.

This Theorem was proved in [4].

**Theorem 2.** Let $X$ be a strictly convex KB-space with a strictly monotone norm and assume that $H$ is an $A_{\alpha, \beta}$-subspace of $C(K, X)$. Then each function $f \in C_\theta(K, X)$ has at most one best $(\alpha, \beta)$-approximant with a weight $\theta \in \Theta$ by elements from $H$ for every $\theta \in \Theta$.

The converse is true if the space $X$ is $(\alpha, \beta)$-smooth. Recall that the space $X$ is $(\alpha, \beta)$-smooth if at every point of its unit sphere there exists a unique tangent functional $\tau_X^{(\alpha, \beta)}(f(x), g(x)) = \tau_{+}^{(\alpha, \beta)}(f(x), g(x))_X = \tau_{-}^{(\alpha, \beta)}(f(x), g(x))_X$. In this case $\tau_X^{(\alpha, \beta)}(f, \cdot)$ is a linear functional.

In this paper, following Strauss and Kroo we show that the converse of Theorem 2.

First we prove the lemma.

**Lemma 1.** Let $X$ be a $(\alpha, \beta)$-smooth KB-space and let $H$ be a finite dimensional subspace $C(K, X)$. For given $h \in H' \setminus \{0\}$ we set $\tilde{H}_h = \{ g \in H : g = 0$ a.e. on $Z_h \}$. Suppose each function $f \in C_\theta(K, X)$ has unique best $(\alpha, \beta)$-approximant with a weight $\theta \in \Theta$ by elements from $H$ for every $\theta \in \Theta$; then for any $h \in H' \setminus \{0\}$ there exists a $g_0 \in \tilde{H}_h$ such that

$$\int_{N_h} \theta(x) \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) \neq 0$$

for any $\theta \in \Theta$.

**Proof.** Suppose each function $f \in C_\theta(K, X)$ has unique best $(\alpha, \beta)$-approximant with a weight $\theta \in \Theta$ by elements from $H$ for every $\theta \in \Theta$. It follows by Theorem 1 for any $h \in H' \setminus \{0\}$ there exists a $g \in H$ such that

$$\int_{N_h} \theta(x) \tau_X^{(\alpha, \beta)}(h(x), g(x)) d\mu(x) > \int_{Z_h} \theta(x) ||g(x)||_{X; \beta, \alpha} d\mu(x),$$

for any $\theta \in \Theta$.

Assume the converse. Then there exists a $h \in H' \setminus \{0\}$ such that for any $g \in \tilde{H}_h$ we can find $\theta \in \Theta$ satisfying

$$\int_{N_h} \theta(x) \tau_X^{(\alpha, \beta)}(h(x), g(x)) d\mu(x) = 0.$$
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Let $g_1, g_2, ..., g_k$ be a basis in $\tilde{H}_b$. Since $X$ is a $(\alpha, \beta)$-smooth KB-space, it follows that the functional $\tau_X^{(\alpha, \beta)}(f, g)$ is linear of $g$ for any fixed $f \in X$, $f \neq 0$. Then for any $\tilde{b} = (b_1b_2, ..., b_k) \in \mathbb{R}^k$ there exists a $\theta_0 \in \Theta$ such that

$$0 = \int_{\tilde{N}_b} \theta_0(x) \tau_X^{(\alpha, \beta)}(h, \sum_{i=1}^k b_i g_i(x))d\mu(x) = \sum_{i=1}^k b_i \int_{\tilde{N}_b} \theta_0(x) \tau_X^{(\alpha, \beta)}(h, g_i(x))d\mu(x). \quad (5)$$

Following Kroo [1] we consider the set

$$A_0 = \left\{ \left( \int_{\tilde{N}_b} \theta(x) \tau_X^{(\alpha, \beta)}(h, g_i(x))d\mu(x) \right)_{i=1}^k : \theta \in \Theta \right\}.$$

$A_0$ is a convex subset of $\mathbb{R}^k$ and $A_0$ has nonempty intersection with any hyperplane $H(\tilde{b}) = \left\{ \tilde{a} \in \mathbb{R}^k : \langle \tilde{a}, \tilde{b} \rangle = 0 \right\}$, $\tilde{b} \in \mathbb{R}^k$, $\langle \tilde{a}, \tilde{b} \rangle$ denotes the inner product in $\mathbb{R}^k$.

Let $A_0$ be an $r$-dimensional convex subset of $\mathbb{R}^k$. Obviously, $\tilde{0}$ is the limit point of $A_0$. Let us show that $\tilde{0} \notin A_0$. If $r = 0$, this is trivial. Now let be $1 \leq r \leq k$ then $A_0$ consists $r$ linearly independent vectors, i.e., there exists $\theta_1, ..., \theta_r \in \Theta$ that

$$\tilde{e}_j = \left( \int_{\tilde{N}_b} \theta_j(x) \tau_X^{(\alpha, \beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^k, \quad 1 \leq j \leq r,$$

are linearly independent.

Show that $A_0$ is an open subset $F_r = \text{span}\{\tilde{e}_1, ..., \tilde{e}_r\}$. Give an arbitrary point $\tilde{c} = \left( \int_{\tilde{N}_b} \theta_i(x) \tau_X^{(\alpha, \beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^k \in A_0 (\theta_i \in \Theta)$ and as well as in [1] indicate ball with center at this point which belongs to $A_0$.

Choosing a sufficiently small $h > 0$, we obtain $\theta_i \pm h\theta_j \in \Theta, 1 \leq j \leq r$ and

$$\tilde{d}_j = \left( \int_{\tilde{N}_b} (\theta_i \pm h\theta_j)(x) \tau_X^{(\alpha, \beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^k = \tilde{c} \pm h\tilde{e}_j \in A_0, 1 \leq j \leq r.$$

Further convexity of $A_0$ implies that for any $\alpha_j^+ \geq 0(1 \leq j \leq r)$ such that

$$\sum_{j=1}^r \alpha_j^+ + \sum_{j=1}^r \alpha_j^- = 1$$

we have $\sum_{j=1}^r \alpha_j^+ \tilde{d}_j^+ + \sum_{j=1}^r \alpha_j^- \tilde{d}_j^- \in A_0$. Then

$$\sum_{j=1}^r \alpha_j^+ \tilde{d}_j^+ + \sum_{j=1}^r \alpha_j^- \tilde{d}_j^- = \tilde{c} + h \sum_{j=1}^r (\alpha_j^+ - \alpha_j^-) \tilde{e}_j \in A_0.$$

Since $\tilde{e}_j, 1 \leq j \leq r$ are linearly independent then we indicated the $r$-dimensional ball with center at the point $\tilde{c}$ from $A_0$ and hence, $A_0$ is the open subset of $F_r$.

Assuming that $\tilde{0} \notin A_0$, we obtain that $\tilde{0} \in \text{Bd}A_0$ and exists a hyperplane $\tilde{F}$ in $F_r$ supporting $A_0$ at $\tilde{0}$. But $A_0$ has nonempty intersection with any hyperplane $H(\tilde{b}) = \left\{ \tilde{a} \in \mathbb{R}^k : \langle \tilde{a}, \tilde{b} \rangle = 0 \right\}$, $\tilde{b} \in \mathbb{R}^k$, $\langle \tilde{a}, \tilde{b} \rangle$ denotes the inner product in $\mathbb{R}^k$.
\{ \tilde{a} \in \mathbb{R}^k : \langle \tilde{a}, \tilde{b} \rangle = 0 \} \text{ and, hence, with } \tilde{F}. \text{ This contradicts the fact that } A_0 \text{ is open in } F_r. \text{ It follows that } \tilde{0} \in A_0 \text{ and exists a weight } \tilde{\theta} \in \Theta \text{ such that }

\int_{N_h} \tilde{\theta}(x) \tau_x^{(\alpha,\beta)}(h(x), g_i(x)) d\mu(x) = 0, \quad 1 \leq i \leq k.

Since \( \tau_x^{(\alpha,\beta)}(h, \cdot)(h \neq 0) \) is the linear functional then

\int_{N_h} \tilde{\theta}(x) \tau_x^{(\alpha,\beta)}(h, g)(x) d\mu(x) = 0, \quad g \in \tilde{H}_h. \quad (6)

Let be \( \dim H = n \) and \( g_1, \ldots, g_n \) is the basis in \( H \) such that the elements \( g_1, \ldots, g_k \) are the basis in \( \tilde{H}_h \) and set \( H_0 = \text{span}\{g_{k+1}, \ldots, g_n\} \). Consider the functionals

\( \eta_1(g) = \int \|g(x)||X;\beta,\alpha d\mu(x), \quad \eta_2(g) = \sup_{x \in K} \|g(x)||X;\alpha,\beta, \quad g \in H_0. \)

These functionals are the norms in \( H_0 \). Since the norms are equivalent in the finite-dimensional space then exists \( \xi > 0 \) such that \( \eta_2(g) \leq \xi \eta_1(g) \) for every \( g \in H_0 \). Let the weight \( \theta^* \in \Theta \) be defined as follows \( \theta^*(x) = \tilde{\theta}(x) \) for \( x \in N_h \) and \( \theta^*(x) = \xi \sup_{x \in K} \tilde{\theta}(x) \mu(K) \) for \( x \in Z_h \).

Then, since the functional \( \tau_x^{(\alpha,\beta)}(h, g) \) is linear, for any \( g = \tilde{g}_1 + \tilde{g}_2 \in H \), where \( \tilde{g}_1 \in \tilde{H}_h, \tilde{g}_2 \in H_0 \), we obtain that

\[
\int_{N_h} \theta^*(x) \tau_x^{(\alpha,\beta)}(h, g)(x) d\mu(x) = \int_{N_h} \tilde{\theta}(x) \tau_x^{(\alpha,\beta)}(h, \tilde{g}_1 + \tilde{g}_2)(x) d\mu(x) = \int_{N_h} \tilde{\theta}(x) \tau_x^{(\alpha,\beta)}(h, \tilde{g}_1 + \tilde{g}_2)(x) d\mu(x) \leq \sup_{x \in K} \tilde{\theta}(x) \mu(K) \eta_2(\tilde{g}_2) \leq \xi \sup_{x \in K} \tilde{\theta}(x) \mu(K) \eta_1(\tilde{g}_2) = \int_{\tilde{Z}_h} \theta^*(x) \|\tilde{g}_2(x)||X;\beta,\alpha d\mu(x) = \int_{\tilde{Z}_h} \theta^*(x) \|g(x)||X;\alpha,\beta d\mu(x).
\]

It contradicts (3).

The lemma is proved.

**Theorem 3.** Let \( X \) be a \( (\alpha, \beta) \)-smooth KB-space, let \( H \) be a finite dimensional subspace \( C(K, X) \) and assume that each function \( f \in C_0(K, X) \) has unique best \( (\alpha, \beta) \)-approximant with a weight \( \theta \in \Theta \) by elements from \( H \) for every \( \theta \in \Theta \). Then \( H \) is the \( A_{\alpha,\beta} \)-subspace of \( C(K, X) \).

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Proof. By Lemma 1, for any $h \in H' \setminus \{0\}$ there exists $g_0 \in \hat{H}_h$ such that
\[
\int_{N_h} \theta(x) \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) \neq 0 \quad \text{for all } \theta \in \Theta.
\]

It means that or $\tau_X^{(\alpha, \beta)}(h(x), g_0(x)) \geq 0$ a.e. on $N_h$, or $\tau_X^{(\alpha, \beta)}(h(x), g_0(x)) \leq 0$ a.e. on $N_h$. Indeed, assuming the contrary, that each set
\[
S_- = \{x \in N_h : \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) < 0\}, \quad S_+ = \{x \in N_h : \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) > 0\}
\]
has positive measure, then
\[
\int_{S_-} \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) < 0,
\]
\[
\int_{S_+} \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) > 0.
\]

For a sufficiently small $\epsilon > 0$ we take the weight functions $\theta_1(x) = 1$ for $x \in S_-$ and $\theta_1(x) = \epsilon$ for $x \in K \setminus S_-$, $\theta_2(x) = 1$ for $x \in S_+$ and $\theta_2(x) = \epsilon$ for $x \in K \setminus S_+$, for which
\[
\int_{N_h} \theta_1(x) \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) < 0,
\]
\[
\int_{N_h} \theta_2(x) \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) > 0.
\]

Then there exists $\gamma \in (0; 1)$ such that
\[
\int_{N_h} (\gamma \theta_1(x) + (1 - \gamma) \theta_2(x)) \tau_X^{(\alpha, \beta)}(h(x), g_0(x)) d\mu(x) = 0,
\]
where $\gamma \theta_1(x) + (1 - \gamma) \theta_2(x) \in \Theta$.

This contradicts the lemma.

Since $\tau_X^{(\alpha, \beta)}$ is linear functional we can assume that $\tau_X^{(\alpha, \beta)}(h(x), g_0(x)) \geq 0$ a.e. at $N_h$.

Thus, we indicate the element $g_0 \in \hat{H}_h$ such that $g_0 = 0$ a.e. on $Z_h$ and $\tau_X^{(\alpha, \beta)}(h(x), g_0(x)) \geq 0$ a.e. at $N_h$. Moreover, by Lemma 1, $\tau_X^{(\alpha, \beta)}(h(x), g_0(x)) > 0$ on a subset of $N_h$ of the positive measure. Hence, set $H$ is the $A_{\alpha, \beta}$-subspace of $C(K, X)$.

The theorem is proved.

Combining the theorems 2 and 3, we get the corollary.

**Corollary 1.** Let $X$ be a strictly convex $(\alpha, \beta)$-smooth KB-space with a strictly monotone norm and let $H$ be a finite dimensional subspace $C(K, X)$. Then each function $f \in C_0(K, X)$ has unique best $(\alpha, \beta)$-approximant with a weight $\theta \in \Theta$ by elements from $H$ for every $\theta \in \Theta$ iff $H$ is the $A_{\alpha, \beta}$-subspace of $C(K, X)$.
These results have been proven in [1] in the case \( \alpha = \beta = 1 \). In the article we use Kroo proof methods of [1].

References


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