General form of \((\lambda, \varphi)\)-additive operators on spaces of \(L\)-space-valued functions

Abstract. The goal of the article is to characterize continuous \((\lambda, \varphi)\)-additive operators acting on measurable bounded functions with values in \(L\)-spaces. As an application, we prove a sharp Ostrowski type inequality for such operators.

Key words: \(L\)-space, \((\lambda, \varphi)\)-additive operator

1. Introduction

In papers [3, 4, 6] questions of numerical analysis and optimal recovery of operators in spaces of functions taking values in semi-linear metric spaces (see [11, 2] for definitions and some properties of \(L\)-spaces) were studied. In [6] the notion of a \((\lambda, \varphi)\)-additive operator was introduced in connection to problems of optimal recovery of monotone operators in partially ordered \(L\)-spaces. In this short paper we will show that any \((\lambda, \varphi)\)-additive operator can be represented as an integral over a vector measure generated by the operator \(\varphi\), and we present some properties of such integrals. The theory of integration with respect to vector measures can be found, for example, in book [8, Chapter IV]. As an application, we obtain one Ostrowski type inequality [10] for such operators.

2. Definitions and notations

2.1. \(L\)-spaces
Definition 1. A set $X$ is called a semilinear space, if operations of addition of elements and their multiplication on real numbers are defined in $X$, and the following conditions are satisfied for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$:

\[
\begin{align*}
x + y &= y + x; \\
x + (y + z) &= (x + y) + z; \\
\exists \theta \in X : x + \theta &= x; \\
\alpha(x + y) &= \alpha x + \alpha y; \\
\alpha(\beta x) &= (\alpha \beta) x; \\
1 \cdot x &= x, \ 0 \cdot x &= \theta.
\end{align*}
\]

Definition 2. We call an element $x \in X$ convex, if for all $\alpha, \beta \geq 0$, \((\alpha + \beta)x = \alpha x + \beta x\). Denote by $X^c$ the subspace of all convex elements of the space $X$.

Definition 3. A semilinear space $X$ endowed with a metric $h_X$ is called an $L$-space, if it is complete and separable and for all $x, y, z \in X$, and $\alpha \in \mathbb{R}$

\[
\begin{align*}
h_X(\alpha x, \alpha y) &= |\alpha|h_X(x, y); \\
h_X(x + z, y + z) &\leq h_X(x, y).
\end{align*}
\]

2.2. Spaces of bounded measurable functions

Let $T$ be a set, $\mathcal{F}_T$ be a $\sigma$-algebra of subsets of $T$. For an $L$-space $X$ denote by $M(T, X)$ the set of all measurable (with respect to the measurable space $(T, \mathcal{F}_T)$) functions $f: T \to X$. It is known that for any $x \in X$ the real-valued function $h_X(x, f(\cdot))$ are measurable if and only if $f \in M(T, X)$ (see e.g. [12, Theorem 1.4.20]). Let $B(T, X)$ be the set of all bounded functions $f \in M(T, X)$. If $f, g \in B(T, X)$, then the function $t \mapsto h_X(f(t), g(t))$ belongs to $B(T, X)$, see e.g [12, Theorem 1.4.22], and in $B(T, X)$ one can define a metric

\[
h_{B(T, X)}(f, g) := \sup_{t \in T} h_X(f(t), g(t)).
\]

We also consider the point-wise partial order in the space $B(T, \mathbb{R})$.

Definition 4. A function $f \in B(T, X)$ is called simple, if it attains a finite set of different values on pairwise disjoint measurable sets i.e., can be represented as

\[
f(t) = \sum_{k=1}^{n} f_k \chi_{T_k}(t), \ t \in T, \ f_k \in X, \ T_k \in \mathcal{F}_T, \ k = 1, \ldots, n; \tag{1}
\]

here and everywhere below, $\chi_A$ is the characteristic function of the set $A$. 
Definition 5. Denote by $B_s(T, X)$ the set of all functions $g \in B(T, X)$ that are limits of uniformly convergent on $T$ sequences of simple functions i.e., $B_s(T, X)$ is the closure in the metric space $(B(T, X), h_{B(T, X)})$ of the set of simple functions.

Separability of $X$ implies that each function $g \in B(T, X)$ is a uniform on $T$ limit of a sequence of piecewise-constant functions with at most countable set of values i.e., functions of the form

$$t \mapsto \sum_{k} f_k \chi_{T_k}(t), \{f_k\} \subset X, \{T_k\} \subset \mathcal{F}_T.$$  

Generally speaking, $B_s(T, X)$ is not guaranteed to coincide with $B(T, X)$, but if $X$ is the set of reals with the usual metric, then $B_s(T, X) = B(T, X)$.

3. $(\lambda, \varphi)$–additive operators

Let $X, Y$ be two $L$–spaces, $(T, \mathcal{F}_T)$ and $(S, \mathcal{F}_S)$ be measurable spaces, $\lambda: X \to Y^c$ be a Lipschitz operator i.e.,

$$h_Y(\lambda x, \lambda y) \leq h_X(x, y) \text{ for all } x, y \in X,$$

and $\varphi: B(T, \mathbb{R}) \to B(S, \mathbb{R})$ be a linear bounded positive operator.

Definition 6. An operator $\Lambda: B_s(T, X) \to B(S, Y)$ is called $(\lambda, \varphi)$–additive, if for arbitrary simple function $f$ with representation (1), one has

$$\Lambda f(s) = \sum_{k=1}^{n} \lambda(f_k) \varphi(\chi_{T_k}).$$

Note that such class of operators was introduced in [4] in connection with some optimal recovery problems. Some important examples of $(\lambda, \varphi)$–additive operators also can be found in [4]. Another example of an $(\lambda, \varphi)$–additive operator can be built as follows.

A surjective operator $P: X \to X^c$ is called a convexifying operator, if it is Lipschitz, $P \circ P = P$ and for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$, $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y)$. Such operators occur in definition of integrals on semi-linear spaces, see e.g. [2, 11].

If $S \subset T$, $X = Y$, and $g \in B(S, \mathbb{R})$ is a fixed non-negative function, then the operator $\Lambda f = (P \circ f) \cdot g$ is $(\lambda, \varphi)$–additive with $\lambda = P$ and $\varphi(w) = w \cdot g$, $w \in B(T, X)$.

In order to find a general representation of $(\lambda, \varphi)$–additive operators, we need the following notion of a vector measure (with values in $B(S, \mathbb{R})$) generated by the operator $\varphi$: for arbitrary set $A \in \mathcal{F}_T$,

$$\mu_\varphi(A) := \varphi(\chi_A).$$
Since $\varphi$ is linear, the obtained vector measure is additive.

For a simple function $f$ of the form (1) we have

$$\Lambda(f) = \sum_{k=1}^{n} \lambda(f_k) \varphi(\chi_{T_k}) =: \int_T \lambda(f(t)) d\mu_\varphi. \quad (2)$$

We show that if $f^n \Rightarrow f$ on $T$, then the sequence $\{\Lambda(f^n)(s)\} = \{\sum_k \lambda(f_k) \varphi(\chi_{T_k})(s)\}$ is fundamental for each $s \in S$. We have

$$h_Y(\Lambda(f^n)(s), \Lambda(f^m)(s)) = h_Y\left( \sum_k \lambda(f^n_k) \varphi(\chi_{T^n_k})(s), \sum_l \lambda(f^m_l) \varphi(\chi_{T^m_l})(s) \right)$$

$$\leq \sum_{k,l} h_Y(\lambda(f^n_k), \lambda(f^m_l)) \varphi(\chi_{T^n_k \cap T^m_l})(s)$$

$$\leq \sum_{k,l} h_X(f^n_k, f^m_l) \varphi(\chi_{T^n_k \cap T^m_l})(s) = \varphi\left( \sum_{k,l} h_X(f^n_k, f^m_l) \chi_{T^n_k \cap T^m_l} \right)(s).$$

It is clear that $\sum_{k,l} h_X(f^n_k, f^m_l) \chi_{T^n_k \cap T^m_l} = h_X(f^n, f^m)$. Hence,

$$h_Y(\Lambda(f^n)(s), \Lambda(f^m)(s)) = h_Y\left( \int_T \lambda(f^n) d\mu_\varphi, \int_T \lambda(f^m) d\mu_\varphi \right)$$

$$\leq \varphi(h_X(f^n, f^m))(s) \Rightarrow 0, \ n, m \to \infty,$$

where the last statement about the uniform convergence on $S$ is due to the uniform convergence of the sequence $\{f^n\}$ and boundedness of $\varphi$.

Thus for each $s \in S$, sequence (2) is fundamental, and since $Y$ is complete, it has a limit. We denote this limit by

$$\int_T \lambda(f) d\mu_\varphi := \lim_{n \to \infty} \int_T \lambda(f^n) d\mu_\varphi, \quad (3)$$

where $f^n \Rightarrow f$ on $T$. It is easy to see that the limit on the right-hand side does not depend on the sequence $\{f^n\}$ (this can be proved using essentially the same arguments as in the proof of the fact that sequence (2) is fundamental), and hence the integral on the left-hand side is well-defined.

**Definition 7.** The integral defined by (3) is called an integral with respect to the vector measure $\mu_\varphi$.

Thus we have proved the following theorem.
Theorem 1. Any continuous \((\lambda,\varphi)\)-additive operator \(\Lambda : B_s(T,X) \to B(S,Y)\) can be represented as
\[
\Lambda(f) = \int_T \lambda(f) d\mu_\varphi,
\]
where the integral on the right-hand side is the integral with respect to the vector measure \(\mu_\varphi\).

Remark 1. The integral from (4) is a function of \(s \in S\). Below for brevity, where it does not lead to an ambiguity, we write \(\int_T \lambda(f) d\mu_\varphi\) instead of \((\int_T \lambda(f) d\mu_\varphi)(s)\).

We need the following lemma, see e.g. [6, Lemma 4].

Lemma 1. For all \(x \in X^c\) and \(\alpha, \beta \in \mathbb{R}\),
\[ h_X(\alpha x, \beta x) \leq |\alpha - \beta| \cdot h_X(x, \theta). \]

The following properties of the introduced integral can easily be established for simple functions and then extended to the case of arbitrary functions \(f \in B_s(T,X)\) by a limiting procedure.

\section{Ostrowski type inequalities}

In this section we obtain some Ostrowski type inequalities for \((\lambda,\varphi)\)-additive operators. Some Ostrowski type inequalities for non-real valued functions can be found in [1, 7, 9, 5].

Let \(T\) be a metric compact with metric \(\rho\), \(\mathcal{F}_T = \mathcal{B}\) be the the \(\sigma\)-algebra of Borel subsets of \(T\). Let also a modulus of continuity \(\omega(t)\) i.e., a non-decreasing continuous semi-additive function such that \(\omega(0) = 0\) be given. Set
\[
H^\omega(T,X) := \{ f : T \to X : \forall t_1, t_2 \in T, h_X(f(t_1), f(t_2)) \leq \omega(\rho(t_1, t_2)) \}. 
\]
Lemma 2. If $T$ is a compact metric space and $\omega$ is a modulus of continuity, then $H^\omega(T, X) \subset B_s(T, X)$.

Proof. Observe that each function $f \in H^\omega(T, X)$ is continuous. Since $T$ is compact, we obtain that $f(T)$ is also compact. Let $n \in \mathbb{N}$ and \{f_1, \ldots, f_m\} $\subset X$, $m \in \mathbb{N}$, be a finite $\frac{1}{n}$-net in $f(T)$. For each $k = 1, \ldots, m$ set

$$T'_k = \left\{ t \in T : h_X(f(t), f_k) \leq \frac{1}{n} \right\} \in \mathcal{F}_T$$

and $T_k = T'_k \setminus \bigcup_{s=1}^{k-1} T'_s$. Then $g_n := \sum_{k=1}^m f_k \chi_{T_k}$ is a simple bounded function and $h_{B(T, X)}(f, g_n) \leq \frac{1}{n}$, which proves the lemma.

We say that an operator $\lambda : X \to Y$ is positively homogeneous, if for arbitrary $\alpha \geq 0$ and $x \in X^c$, $\lambda(\alpha x) = \alpha \lambda(x)$. For a positively homogeneous operator $\lambda$ one has

$$\lambda(\theta_X) = \lambda(0 \cdot \theta_X) = 0 \cdot \lambda(\theta_X) = \theta_Y.$$ 

A sequence \{x_n\} $\subset X$ is called $\lambda$-extremal, if $h(x_n, \theta_X) = 1$ for all $n \in \mathbb{N}$, and $h_Y(\lambda(x_n), \theta_Y) \to 1$, as $n \to \infty$.

Theorem 2. Let $\Lambda : B_s(T, X) \to B(S, Y)$ be a continuous $(\lambda, \varphi)$-additive operator, and $\omega$ be a modulus of continuity. Then for arbitrary function $f \in H^\omega(T, X)$ and arbitrary $t_0 \in T$ the following inequality holds

$$h_Y(\Lambda(f), \Lambda(f(t_0) \chi_T)) \leq \int_T \omega(\rho(t, t_0))d\mu_\varphi. \tag{5}$$

If additionally the operator $\lambda$ is positively homogeneous and there exists a $\lambda$-extremal sequence \{x_n\} $\subset X^c$, then inequality (5) is sharp.

Proof. Taking into account Theorem 1, property P2, the fact that $\lambda$ is a Lipschitz operator, $f \in H^\omega(T, X)$, and property P3 we obtain

$$h_Y(\Lambda(f), \Lambda(f(t_0) \chi_T)) = h_Y \left( \int_T \lambda(f) d\mu_\varphi, \int_T \lambda(f(t_0) \chi_T) d\mu_\varphi \right) \leq \int_T h_Y(\lambda(f), \lambda(f(t_0) \chi_T)) d\mu_\varphi \leq \int_T h_X(f, f(t_0) \chi_T) d\mu_\varphi \leq \int_T \omega(\rho(t_0, t))d\mu_\varphi.$$ 

Inequality (5) is proved.

Assume $\lambda$ is positively homogeneous and \{x_n\} $\subset X^c$ is a $\lambda$-extremal sequence. For each $n \in \mathbb{N}$, set

$$f_e(t) = f^n_e(t) = x_n \cdot \omega(\rho(t, t_0)).$$

Then for arbitrary $t_1, t_2 \in T$, using Lemma 1 we obtain

$$h_X(f_e(t_1), f_e(t_2)) = h_X(x_n \cdot \omega(\rho(t_1, t_0)), x_n \cdot \omega(\rho(t_1, t_0))) \leq |\omega(\rho(t_1, t_0)) - \omega(\rho(t_2, t_0))| h_X(x_n, \theta) \leq \omega(|\rho(t_1, t_0) - \rho(t_2, t_0)|) \leq \omega(\rho(t_1, t_2)),$$
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and hence \(f_e \in H^\omega(T, X)\). Besides that, using property \(P1\) of the integral, we have for each \(s \in S\) (which we omit for brevity)

\[
\begin{align*}
    h_Y(\Lambda(f_e), \Lambda(f_e(t_0)\chi_T)) &= h_Y \left( \int_T \lambda(f_e) d\mu_\varphi, \int_T \lambda(f_e(t_0)\chi_T) d\mu_\varphi \right) \\
    &= h_Y \left( \lambda(x_n) \int_T \omega(\rho(t, t_0)) d\mu_\varphi, \theta_Y \right) = h_Y (\lambda(x_n), \theta_Y) \int_T \omega(\rho(t, t_0)) d\mu_\varphi \\
    &\to \int_T \omega(\rho(t, t_0)) d\mu_\varphi \text{ as } n \to \infty,
\end{align*}
\]

which proves sharpness of inequality (5). The theorem is proved.

References


Received: 21.05.2022. Accepted: 22.06.2022